22. 51 -- Interaction of Radiation With Matter

Home Work No. 4

1. Consider a mass point going around the origin in a circular orbit on the x-y plane. Denoting the azimuthal angle operator by $\hat{\varphi}$ and z-component of the orbital angular momentum operator by $\hat{L}_z$. Show that the following commutation relation is valid for these two operators,

$$\left[\hat{\varphi}, \hat{L}_z\right] = i\hbar \tag{1}$$

Derive an uncertainty relation from the above commutation relation. Explain its physical implication and give a useful relation you can think of from this uncertain relation.

2. Construct the matrix representation of a spin 1/2 system. If j=1/2, then there are two possible m values, namely, $m = \pm 1/2$. Denote the two eigenkets $|j,m\rangle$ by column vectors as:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{2}$$

show that the matrix representations of the relevant operators are given by:

$$\hat{J}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{J}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{J}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{J}^2 = \frac{3\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

Do the same thing for the spin 1 system, where the three eigenkets are represented as:

$$|1,1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |1,0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |1,-1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4}$$

Define the Pauli matrices as:

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{5}$$

proof that they have the following properties:

(a) $\left[\hat{\sigma}_i, \hat{\sigma}_j\right] = 2i\hat{\sigma}_k$, where i, j, k form an even permutation of x, y, and z. (6)

(b) $\left[\hat{\sigma}_i, \hat{\sigma}_j\right]_+ \equiv \hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{ij}$, where i, j = x, y, or z. (7)

(c) Define $\hat{\sigma}_+ = \frac{1}{2}\left(\hat{\sigma}_x + i\hat{\sigma}_y\right)$ and $\hat{\sigma}_- = \frac{1}{2}\left(\hat{\sigma}_x - i\hat{\sigma}_y\right)$, then (8)

$$\left(\hat{\sigma}_+\hat{\sigma}_-\right)^n = \hat{\sigma}_+\hat{\sigma}_- = \frac{1}{2}\left(\hat{\sigma}_z + 1\right). \tag{9}$$

(d) If ξ is a parameter, then

$$e^{1/2\xi\hat{\sigma}_z}\hat{\sigma}_+ e^{-1/2\xi\hat{\sigma}_z} = \hat{\sigma}_+ e^{\xi}$$

and (10)

$$e^{1/2\xi\hat{\sigma}_z}\hat{\sigma}_- e^{-1/2\xi\hat{\sigma}_z} = \hat{\sigma}_- e^{-\xi}$$

(e) From the above results it follows that

$$e^{1/2\xi\hat{\sigma}_z}\hat{\sigma}_x e^{-1/2\xi\hat{\sigma}_z} = \hat{\sigma}_x \cosh\frac{1}{2}\xi + i\hat{\sigma}_y \sinh\frac{1}{2}\xi$$

and (11)

$$e^{1/2\xi\hat{\sigma}_z}\hat{\sigma}_y e^{-1/2\xi\hat{\sigma}_z} = \hat{\sigma}_y \cosh\frac{1}{2}\xi - i\hat{\sigma}_x \sinh\frac{1}{2}\xi$$

3. Given a harmonic oscillator Hamiltonian,

$$H = \frac{1}{2}\hat{P}^2 + \frac{1}{2}\omega^2\hat{Q}^2$$

we can construct an operator U(p,q) such that

$$U(p,q) = \exp[\frac{i}{\hbar}(p\hat{Q} - q\hat{P})] \tag{12}$$

in which p and q are complex numbers. Starting from the ground state of the oscillator $|0>$, one can generate a state $|p,q>$ by:

$$|p,q> = U(p,q)\ |0> . \tag{13}$$

The objective of this problem is to establish that the state $|p,q>$ has all the properties of the coherent state $|\alpha>$.

(a) Prove an operator identity

$$e^{\hat{Y}}\hat{X}e^{-\hat{Y}} = \hat{X} + \left[\hat{Y},\hat{X}\right] + \frac{1}{2!}\left[\hat{Y},\left[\hat{Y},\hat{X}\right]\right] + \cdots \tag{14}$$

(b) Use (7) to prove that

$$U^{-1}(p,q)\left[\alpha\hat{P} + \beta\hat{Q}\right]U(p,q) = \alpha\left(\hat{P} + p\right) + \beta\left(\hat{Q} + q\right) \tag{15}$$

so that U(p,q) is a translational operator.

(c) Consider a modified Hamiltonian $H' = \frac{1}{2}\hat{P}^2 + \frac{1}{2}\omega^2\hat{Q}^2 - \frac{1}{2}\hbar\omega$ and show that

$$H'\ |0> = 0 \tag{16}$$

namely that the ground state $|0>$ is an eigen state of H' with an eigen value zero.

(d) Prove that

$$0 = U(p,q)H'|0> = UH'U^{-1}U|0> = UH'U^{-1}|p,q> = \left\{\frac{1}{2}(\hat{P} - p)^2 + \frac{1}{2}\omega^2(\hat{Q} - q)^2 - \frac{1}{2}\hbar\omega\right\}|p,q> . \tag{17}$$

This means that $|p,q>$ is an eigen state of the harmonic oscillator Hamiltonian in which the coordinate is displaced by q and the momentum by p.

(e) Prove also that the mean values of the position and momentum in $|p,q>$ state are q and p respectively, i.e.

$$\begin{aligned}\langle p,q|\hat{Q}|p,q\rangle &= q \\ \langle p,q|\hat{P}|p,q\rangle &= p\end{aligned} \tag{18}$$

(f) Show that |p,q> is the coherent state |α> with α given by

$$\alpha = \sqrt{\frac{\omega}{2\hbar}}\left[q + \frac{ip}{\omega}\right] \tag{19}$$

Hint: the coherent state |α> is generated by (class notes)

$$|\alpha\rangle = \exp\left[\alpha a^{+} - \alpha^{*} a\right]0\rangle \tag{20}$$