MIT 8.03 Fall 2004 – Solutions to Problem Set 6

Problem 6.1 — Phase and group velocity in your bathtub

The dispersion relation for deep-water waves is given approximately by

$$\omega^2 = gk + \frac{T}{\rho}k^3,$$

where $\omega = 2\pi/\lambda$.

(a) For very short wavelengths ($\lambda \ll 1.7$ cm), the k^3 term dominates. Then $\omega^2 \approx T/\rho k^3$. Then, the phase velocity is

$$v_p = \frac{\omega}{k} = \sqrt{\frac{Tk}{\rho}}.$$

The group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{Tk}{\rho}}.$$

Combining these two equations gives $v_g = 3/2 v_p$.

(b) For very long wavelengths ($\lambda \gg 1.7$ cm), the k term dominates. Then $\omega^2 \approx gk$. Then, the phase velocity is

$$v_p = \sqrt{\frac{g}{k}}.$$

And the group velocity is

$$v_g = \frac{1}{2}\sqrt{\frac{g}{k}}.$$

Hence, $v_g = v_p/2$.

Problem 6.2 — Shallow-water waves (Home experiment)

This experiment was performed by Igor Sylvester.

(a) I made many measurements and finally concluded that it took about 3 s for a wave packet to travel 4 times the diameter (23 cm) of a pan with a depth of 9 mm. The uncertainty in this is about 0.5 s (17% error). Even though I used a stopwatch that can measure time with an accuracy of 10 ms, the error in my measurement is much larger because it's not easy to tell precisely where the packet is.

(b) The speed of the wave packet based on my results is 31 ± 5 cm/s. This is in good agreement with the predicted value of 29.7 cm/s.

Problem 6.3 — (French 7–20) Why are deep-water waves dispersive?

(a) The potential energy of the liquid is

$$U = mgh = (\rho Ay)gy = \rho Agy^2.$$

The kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(\rho Al)\left(\frac{dy}{dt}\right)^2.$$

Then, we can derive the equation of motion from conservation of energy:

$$\begin{aligned}\frac{\partial E}{\partial t} &= 0 \\ &= (2A\rho g y + \rho A l \ddot{y})\dot{y} \\ \Rightarrow \ddot{y} + \frac{2g}{l} y &= 0.\end{aligned}$$

This is a simple harmonic oscillator. Hence, the period of oscillations is $T = \pi\sqrt{2l/g}$.

(b) We know that $v = \nu\lambda$. Assuming that $\lambda \approx 2l$, $v = 2\nu l = \sqrt{g\lambda}/\pi$.

(c) For $\lambda = 500$ m, $v = 27$ m/s ≈ 97 km/h ≈ 61 mi/h.

Problem 6.4 — Energy in waves

(a) Equation 7–38 in French gives the energy per wavelength in a traveling wave. Using $v = \sqrt{T/\mu}$ and $\nu = v\lambda$, eq. 7–38 is

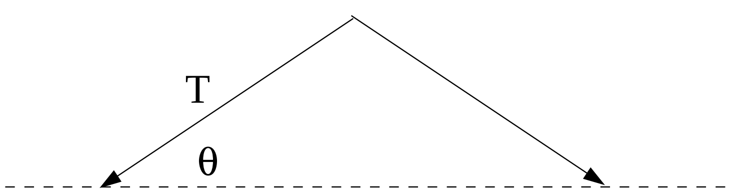
$$W_{\text{cycle}} = 2\pi^2\nu^2A^2\lambda\mu = 2\pi^2A^2\frac{T}{\lambda}.$$

The equation

$$E_\lambda = \frac{\pi^2A^2T}{\lambda}$$

is the energy stored in one wavelength of a standing wave. Note that $W_{\text{cycle}} = 2E_\lambda$. This is correct because the energy per wavelength in a traveling wave is double that of a standing wave (same amplitude).

(b) The following graph shows the deformed string (highly exaggerated).



If the tension remains approximately constant (for modest distortion) then the work needed to pick up the string is

$$W = \int_0^A F(y)\,dy,$$

where

$$\begin{aligned}F(y) &= 2T\sin\theta \\ &\approx 2T\frac{y}{L/2} \\ &= \frac{4T}{L}y.\end{aligned}$$

Then,

$$W = \int_0^A \left(\frac{4T}{L}\right) y\,dy = \frac{2TA^2}{2}.$$

(c)

$$\begin{aligned} W_{\mathrm{TOT}} &= nW \\ &= n\int_0^{A_n}\left(\frac{2T}{L/2n}\right) y\,dy \\ &= \frac{2Tn^2A_n^2}{L}. \end{aligned}$$

(d) For the triangular wave, $L = n\lambda/2$ and

$$\begin{aligned} E_{\lambda_{\mathrm{triangle}}} &= \frac{2}{n}\frac{2Tn^2A_n^2}{n\lambda/2} \\ &= \frac{8TA_n^2}{\lambda}. \end{aligned}$$

Then, the energy ratio is

$$\begin{aligned} \frac{E_{\lambda_{\mathrm{sine}}}}{E_{\lambda_{\mathrm{triangle}}}} &= \frac{\pi^2}{8} \\ &\approx 1.25. \end{aligned}$$

Problem 6.5 — Energy in traveling waves on a string

(a) A standing wave with amplitude A can be created by two traveling waves, moving in opposite directions, each with amplitude $0.5A$. Thus, the total energy (per wavelength λ) is half that of the standing wave with amplitude A. When the standing wave stands still, all it energy is in the form of potential energy, which is proportional to A^2. For one of the two traveling waves (amplitude $0.5A$), the potential energy is proportional to $A^2/4$ and it is independent of time. Thus, its kinetic energy (at any moment in time) must also be $A^2/4$, so that its total energy per wavelength is half that of the standing wave.

(b) Let's calculate the kinetic and potential energies in one wavelength explicitly. The wave is $y(x,t) = A\sin(\omega t - kx)$, where $k = 2\pi/\lambda$, $\omega = vk$ and $v^2 = T/\mu$. The kinetic energy is

$$\begin{aligned} K &= \int_0^{\lambda}\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 dx \\ &= \frac{\mu}{2}\int_0^{\lambda} A^2\omega^2\cos^2(\omega t - kx)\,dx \\ &= \frac{\mu A^2\omega^2}{2}\frac{\lambda}{2} \\ &= \frac{TA^2\pi^2}{\lambda}. \end{aligned}$$

The potential energy is

$$\begin{aligned} U &= \int_0^{\lambda}\frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx \\ &= \frac{T}{2}\int_0^{\lambda} A^2k^2\cos^2(\omega t - kx)\,dx \\ &= \frac{TA^2k^2}{2}\frac{\lambda}{2} \\ &= \frac{TA^2\pi^2}{\lambda}. \end{aligned}$$

As expected, the kinetic and potential energies are equal. The total energy in one wavelength of a traveling wave is $2TA^2\pi^2/\lambda$.

Problem 6.6 — (Bekefi & Barrett 3.3) Electromagnetic plane waves

(a) First note that $\vec{B} = B_y\hat{y}$. Hence, $B_x = B_z = 0$. We now proceed by applying Maxwell's equations to $\vec{B}$ and $\vec{E}$. Gauss' Law for electricity states that

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 0 \\
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= 0 \\
E_{0_x} f' k_x + E_{0_y} f' k_y + E_{0_z} f' k_z &= 0 \\
E_{0_x} k_x + E_{0_y} k_y + E_{0_z} k_z &= 0 \\
\Rightarrow \vec{E}\cdot\vec{k} &= 0.
\end{aligned}$$

Similarly, Gauss' Law for magnetism gives $\vec{B}\cdot\vec{k} = 0$. Ampérè's Law says that

$$\begin{aligned}
\vec{\nabla}\times\vec{B} &= \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \\
\frac{\partial B_y}{\partial x}\hat{z} - \frac{\partial B_y}{\partial z}\hat{x} &= \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \\
k_x B_{0_y} f'\hat{z} - k_z B_{0_y} f'\hat{x} &= \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}.
\end{aligned}$$

Note here that $E_y = 0$ since the left side of the equation does not have a component along $\hat{y}$. Integrating the former equation with respect to t gives

$$\begin{aligned}
\vec{E} &= \frac{c^2 B_{0_y}}{\omega}\left(k_x f\hat{z} - k_z f\hat{x}\right) + \vec{C}(\vec{r}) \\
&= B_{0_y}\frac{c^2}{\omega} f\cdot\left(k_x\hat{z} - k_z\hat{x}\right) + \vec{C}(\vec{r}),
\end{aligned}$$

where $\vec{C}(\vec{r})$ is a constant of integration. You can quickly check that $\vec{\nabla}\cdot\vec{E} = 0$ implies $\vec{\nabla}\cdot\vec{C} = 0$. Also, we can use Faraday's Law to show that $\vec{\nabla}\times\vec{C} = 0$. The details of the algebraic steps are left as an exercise to the reader. It turns out that $\vec{\nabla}\cdot\vec{C} = 0$ and $\vec{\nabla}\times\vec{C} = 0$ imply $\vec{C} = 0$. Then, using $\omega = |k|c$,

$$\begin{aligned}
\vec{E} &= B_{0_y}\frac{c}{|k|} f\cdot\left(k_x\hat{z} - k_z\hat{x}\right) \\
&= B_{0_y} c\left(\frac{k_x}{|k|} f\hat{z} - \frac{k_z}{|k|} f\hat{x}\right) \\
\vec{E} &= -c\,\hat{k}\times\vec{B}.
\end{aligned}$$

Consequently, $|\vec{E}| = c|\vec{B}|$ and $\hat{E}\times\hat{B} = \hat{k}$. Thus, $\vec{E}\perp\vec{k}$ and $\vec{E}\perp\vec{B}$. Note that the direction of propagation of the wave, $\hat{k}$, equals the direction of the Poynting vector $\vec{S}$.

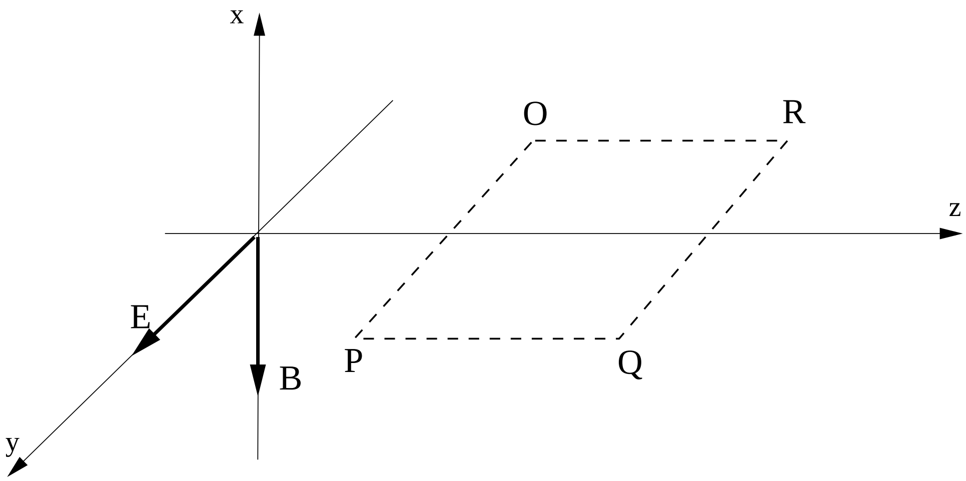
(b) If $k_z = 0$ then $\vec{k} = k_x$ and $\vec{E} = E_z\hat{z}$ since $\vec{k}\perp\vec{B}\perp\vec{E}$. Using the equation derived in the previous section

$$\begin{aligned}
\vec{E} &= -c\hat{k}\times\vec{B} \\
&= cB_{0_z} f\left(\vec{k}\cdot\vec{r} - \omega t + \phi\right)\vec{z}.
\end{aligned}$$

Problem 6.7 — (Bekefi & Barrett 3.5) Maxwell in action

(a) Since $\vec{B} = B_0 \sin(\omega t - kz)\hat{x}$, $\hat{k} = \hat{z}$. Using $\vec{E} = -c\hat{k} \times \vec{B}$, $\vec{E} = -cB_0 \sin(\omega t - kz)\hat{y}$.

(b) The following sketch shows the values of $\vec{E}$ and $\vec{B}$ at the origin and the wire loop.



The EMF in the loop is given by

$$\begin{aligned}\mathcal{E} &= \oint \vec{E} \cdot d\vec{l} \\ &= -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}.\end{aligned}$$

We choose the surface of integration S to be the flat square bounded by the loop. Then, $d\vec{S}$ is normal to that surface. Since we will call positive $\mathcal{E}$ if it is in the direction $QPORQ$, then $\hat{S} = -\hat{x}$ by the right-hand rule. Furthermore, since the electromagnetic field is a plane wave, this problem has translational symmetry. Then, for convenience, we will choose the coordinate system such that the origin is at point O of the loop. Then,

$$\begin{aligned}\mathcal{E} &= -\frac{d}{dt} \int_0^\lambda \int_0^\lambda B_0 \sin(\omega t - kz)\hat{x} \cdot (\hat{x}\, dydz) \\ &= -B_0\omega \int_0^\lambda \int_0^\lambda \cos(\omega t - kz)\, dydz.\end{aligned}$$

Using $k = 2\pi/\lambda$,

$$\begin{aligned}\mathcal{E} &= -\lambda B_0 \omega\, \frac{\lambda}{2\pi} \sin\left(\omega t - \frac{2\pi}{\lambda} z\right)\Bigg|_{z=0}^{z=\lambda} \\ &= 0.\end{aligned}$$

Alternatively, we can calculate the integral using the electric field of the electromagnetic wave. Notice that the path of integration is $QPORQ$.

$$\begin{aligned}\mathcal{E} &= \oint \vec{E}\cdot d\vec{l} \\ &= \underbrace{\int_Q^P \vec{E}\cdot d\vec{l}}_{=0} + \int_P^O \vec{E}\cdot d\vec{l} + \underbrace{\int_O^R \vec{E}\cdot d\vec{l}}_{=0} + \int_R^Q \vec{E}\cdot d\vec{l} \\ &= \int_P^O \vec{E}\cdot d\vec{l} + \int_R^Q \vec{E}\cdot d\vec{l}.\end{aligned}$$

Since the loop has sides of length λ, the electric field along PO equals the electric field along RQ. Mathematically, $\vec{E}(z_{PO}) = E_0 \sin(\omega t - k z_{PO})\hat{y}$ and $\vec{E}(z_{RQ}) = E_0 \sin(\omega t - k z_{RQ})\hat{y}$. Since $z_{RQ} = z_{PO} + \lambda$, $\vec{E}(z_{RQ}) = E_0 \sin(\omega t - k z_{PO} - 2\pi)\hat{y} = E_0 \sin(\omega t - k z_{PO})\hat{y} = \vec{E}(z_{PO})$. Hence, $\int_P^O \vec{E}\cdot d\vec{l} = -\int_R^Q \vec{E}\cdot d\vec{l}$. Therefore, $\mathcal{E} = 0$ along the loop.

(c) Rotating the loop about the z-axis leaves $\mathcal{E}_I = 0$; the argument is similar to the one in part (a). Similarly, rotating the loop about the x-axis still leaves $\mathcal{E}_I = 0$. However, rotating the loop about the y-axis changes the value of $\mathcal{E}_I$. We can calculate the inclination of the plane which maximizes $\mathcal{E}$. The EMF in the loop is defined as

$$\mathcal{E} = -\int_S \frac{d\vec{B}}{dt}\cdot d\vec{S}.$$

Hence, maximizing $|\mathcal{E}|$ implies maximizing $\left|\oint_S \frac{d\vec{B}}{dt}\cdot d\vec{S}\right|$. If the loop is flat against the YZ plane then the net flux of $\frac{\partial\vec{B}}{\partial t}$ through the plane of the loop is zero (positive flux cancels cancels an equal amount of negative flux). We wish that only positive (or only negative) flux crosses the plane of the loop. Hence, the loop must be oriented so that its projected area onto the YZ plane is half of its area. In other words, $\vec{A}\cdot\hat{x} = A/2$, where $\vec{A} = A\hat{\text{n}}$, A is the area of the loop and $\hat{\text{n}}$ is a unit vector normal to the surface of the loop. Then, $\vec{A}\cdot\hat{x} = A\cos\theta = A/2$ and $\theta = \cos^{-1}(1/2) = \pi/3 = 60^o$. Hence, the EMF is

$$\begin{aligned}\mathcal{E} &= -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S} \\ &= -\frac{d}{dt}\int_0^{\lambda\cos\pi/3}\int_0^{\lambda} B_0\sin(\omega t - kz)\,dydz \\ &= -B_0\omega\lambda\frac{\lambda}{\pi}\sin\left(\omega t - \frac{2\pi}{\lambda}z\right)\Bigg|_{z=0}^{z=\lambda\cos\pi/3} \\ &= 2\lambda B_0 c\cos\omega t.\end{aligned}$$

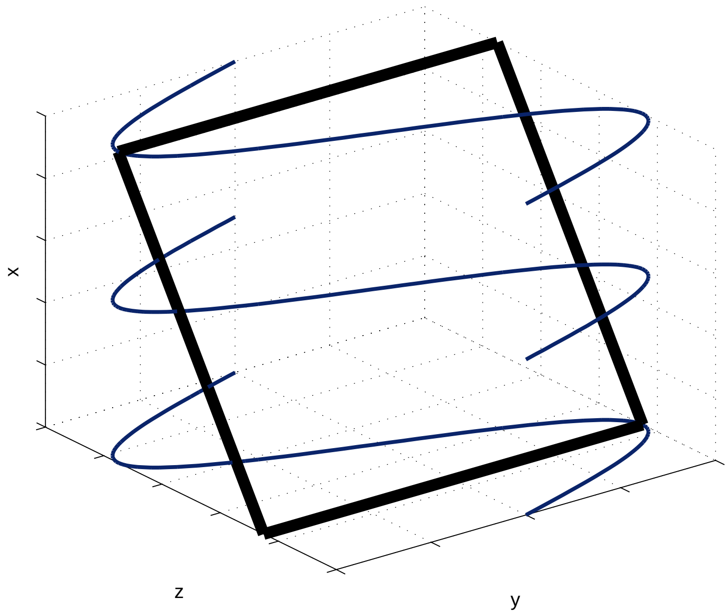
So the maximum EMF is

$$\mathcal{E}_0 = 2\lambda B_0 c = 2E_0.$$

Alternatively, we can use Faraday's Law

$$\mathcal{E} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} = \oint \vec{E}\cdot d\vec{l}.$$

Then, maximizing $|\mathcal{E}|$ implies maximizing $|\oint_L \vec{E}\cdot d\vec{l}|$. The following sketch shows the $\vec{E}$ field (blue lines) and the wire loop (thick black line) oriented at 60^o with respect to the YZ plane. Note that this orientation of the loop gives the maximum $\oint_L \vec{E}\cdot d\vec{l}$.



Note that whether you calculate $-\frac{d\Phi_B}{dt}$ using only the magnetic field or $\oint \vec{E} \cdot d\vec{l}$ using only the electric field of the EM wave you will find the same result. You should not think of the electric and magnetic fields of an EM wave as being independent. Instead, you should remember that the B-field causes the E-field and the E-field causes the B-field. They are "one and the same."