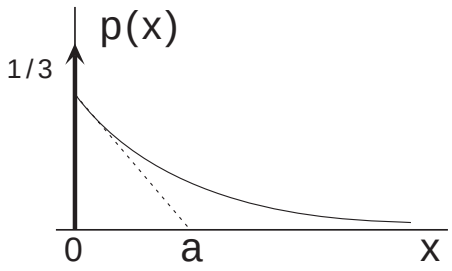
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I Spring Term 2003

Practice Exam #1

Problem 1 (35 points) Clearing Impurities



In an effort to clear impurities from a fabricated nano-wire a laser beam is swept repeatedly along the wire in the presence of a parallel electric field. After one sweep an impurity initially at $x = 0$ has the following probability density of being found at a new position x

$$p(x) = \frac{1}{3}\,\delta(x) + \frac{2}{3a}\exp[-x/a] \qquad 0 \le x$$

$$= 0 \qquad \text{elsewhere}$$

where a is some characteristic length.

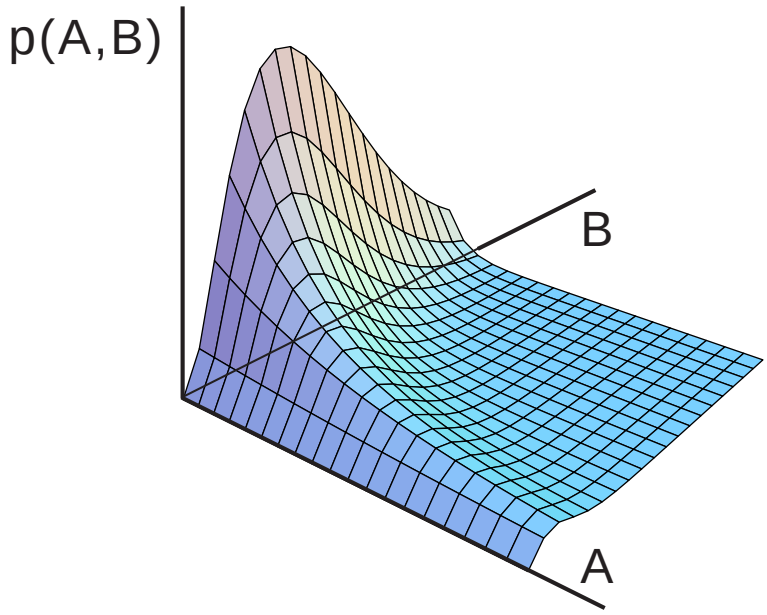
a) Find the cumulative function $P(x)$. Make a sketch of the result which displays all of its important features.

b) What is the probability that x will be displaced by at least an amount a by a single sweep of the laser beam?

c) Find the mean and the variance of x in terms of a.

d) Give an approximate probability density for the total distance d the impurity has moved along the wire after 36 sweeps of the laser beam.

Problem 2 (35 points) Polarization of the 21 cm Line



A radio astronomer is studying the polarization of the 21 cm line emitted by a cloud of interstellar hydrogen gas. She finds that the joint probability density for the intensity of the radiation polarized parallel, A, and perpendicular, B, to some physically significant direction is given by

$$\begin{aligned} p(A,B) &= \frac{\gamma^2 B^2}{\alpha^6}\,\exp[-B(A+\gamma)/\alpha^2] && 0 \le A, B \\ &= 0 && \text{elsewhere} \end{aligned}$$

α and γ are parameters with the units of intensity.

a) Find $p(A)$ and $p(B)$. Sketch the results.

b) Find the conditional probability density $p(A\,|B)$. Sketch the result.

c) Are A and B statistically independent? Explain your reasoning.

Problem 3 (30 points) Quotient of Random Numbers

The statistically independent random variables x and y are each uniformly distributed in the interval between 0 and 1. Find the probability density $p(q)$ for the quotient $q \equiv x/y$. Make a carefully labeled sketch of your result. [Note: if your answer does not come out normalized, you have made a mistake.]

Integrals The next page contains a number of indefinite integrals. Some potentially useful definite integrals are given below.

For integer n and m

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx = \sqrt{\pi}$$

$$(2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2\sigma^2}\, dx = 1 \cdot 3 \cdot 5 \cdots (2n-1)\, \sigma^n$$

$$\int_0^\infty x\, e^{-x^2}\, dx = \frac{1}{2}$$

$$\int_0^1 x^m (1-x)^n\, dx = \frac{n!m!}{(m+n+1)!}$$