8.044 SOLUTIONS EXAM #1

1. a)

b) $\langle f \rangle = 2 f_0 \left(\frac{1}{2}\right) + f_0 \left(\frac{1}{3}\right) + 4 f_0 \left(\frac{1}{6}\right) = f_0 \left(\frac{3}{3} + \frac{1}{3} + \frac{2}{3}\right) = \underline{\underline{2 f_0}}$

c) $\langle f^2 \rangle = 4 f_0^2 \left(\frac{1}{2}\right) + f_0^2 \left(\frac{1}{3}\right) + 16 f_0^2 \left(\frac{1}{6}\right) = f_0^2 \left(\frac{6}{3} + \frac{1}{3} + \frac{8}{3}\right) = 5 f_0^2$

$$\mathrm{Var}(f) = \langle f^2 \rangle - \langle f \rangle^2 = (5-4) f_0^2 = \underline{\underline{f_0^2}}$$

d) ALL BUT ONE ISOTOPE MUST HAVE $A = n$.
THE OTHER (64 CHOICES) MUST HAVE $A = n+1$

$$\Rightarrow p\big(M = (64n+1) m_0\big) = \underline{\underline{64 \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{63}}}$$

e) USE THE CENTRAL LIMIT THEOREM

$$p(M) \approx \left(\frac{m_0}{\sqrt{2\pi \,\mathrm{Var}(M)}} \, e^{-\frac{(M - \langle M \rangle)^2}{2\,\mathrm{Var}(M)}} \right) \cdot \sum_{i=1}^{\infty} \delta(M - i m_0)$$

WHERE $\langle M \rangle = 64\, m_0 \langle A \rangle$

$\mathrm{Var}(M) = 64\, m_0^2\, \mathrm{Var}(A)$

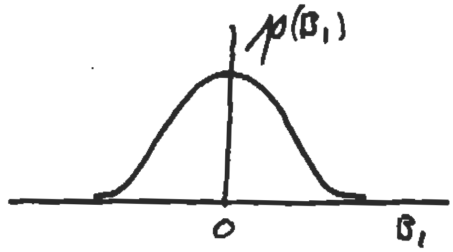
2. a) $$p(B_1) = \int_{-\infty}^{\infty} p(B_1, B_2)\, dB_2 = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t/\tau} e^{-B_1^2/2\sigma^2}$$

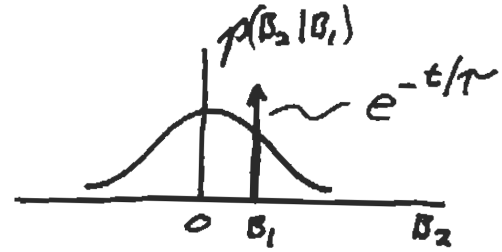
$$+ \frac{1}{\sqrt{2\pi\sigma^2}} \left(1 - e^{-t/\tau}\right) e^{-B_1^2/2\sigma^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-B_2^2/2\sigma^2}\, dB_2}_{1}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-B_1^2/2\sigma^2}$$

NOTE THAT THIS DOES NOT DEPEND ON τ

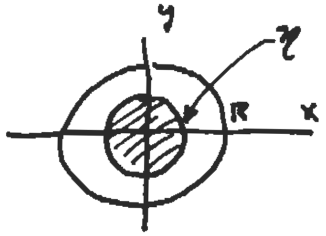
b) $$p(B_2 \mid B_1) = \frac{p(B_1, B_2)}{p(B_1)} = e^{-t/\tau} \delta(B_2 - B_1) + \left(1 - e^{-t/\tau}\right) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-B_2^2/2\sigma^2}$$





c) B_1 & B_2 ARE NOT S.I. BECAUSE $p(B_1, B_2) \neq p(B_1)\, p(B_2)$

NOTE THAT $p(B_1, B_2)$ DEPENDS ON τ BUT $p(B_1)$ AND $p(B_2)$ DO NOT.

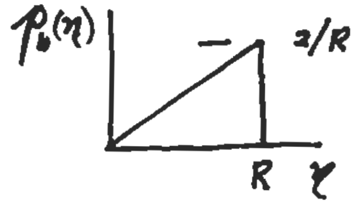
3. a) 

$b < \eta$ IN SHADED REGION

$$P_b(\eta) = (\pi \eta^2)\left(\frac{1}{\pi R^2}\right) = \eta^2/R^2$$

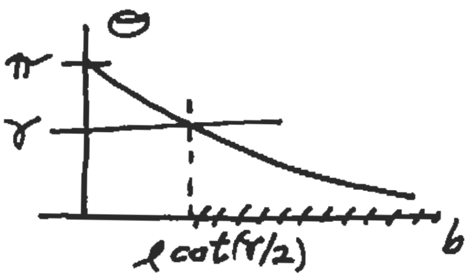
$$p_b(\eta) = \frac{d P_b(\eta)}{d\eta} = 2\eta/R^2 \quad 0 < \eta < R$$

$$= 0 \quad \text{ELSEWARE}$$



b) LARGE $b \rightarrow$ SMALL Θ

LARGEST $b = R \Rightarrow \Theta_{MIN} = 2\,\mathrm{ARCCOT}(R/\ell)$

c) 

$$P_\Theta(\gamma) = \int_{\ell \cot(\gamma/2)}^{\infty} p(b)\, db$$

$$p_\Theta(\gamma) = \frac{d}{d\gamma} P_\Theta(\gamma) = (-1)\left(-\frac{\ell}{2}\csc^2(\gamma/2)\right)\left(2\,\ell\cot(\gamma/2)/R^2\right)$$

$$= \frac{\ell^2}{R^2}\,\frac{1}{\sin^2(\gamma/2)}\,\frac{\cos(\gamma/2)}{\sin(\gamma/2)}$$

$$= \frac{\ell^2}{R^2}\,\frac{\cos(\gamma/2)}{\sin^3(\gamma/2)} \qquad \Theta_{MIN} \leq \gamma \leq \pi$$

$$= 0 \quad \text{ELSEWARE}$$