

Problem Set #12

Solutions Available in on the Web: Wednesday, May 12

Problem 1: Two Identical Particles

A system consists of two identical, non-interacting, spinless (no spin variables at all) particles. The system has only three single-particle states ψ_1 , ψ_2 , and ψ_3 with energies $\epsilon_1 = 0 < \epsilon_2 < \epsilon_3$ respectively.

- a) List in a vertical column all the two-particle states available to the system, along with their energies, if the particles are Fermions. Use the occupation number notation (n_1, n_2, n_3) to identify each state. Indicate which state is occupied at $T = 0$.
- b) Repeat a) for the case of Bose particles.
- c) Use the Canonical Ensemble to write the partition function for both Fermi and Bose cases.
- d) Using only the leading two terms in the partition function, find the temperature dependence of the internal energy in each case. Contrast the behavior of the internal energy near $T = 0$ in the two cases.

Problem 2: A Number of Two-State Particles

Consider a collection of N non-interacting, spinless Bose particles. There are only two single-particle energy eigenstates: ψ_0 with energy $\epsilon = 0$ and ψ_1 with energy $\epsilon = \Delta$.

- a) How would you index the possible N -body energy eigenstates in the occupation number representation? What are their energies? How many N -body states are there in all?
- b) Find a closed form expression for the partition function $Z(N, T)$ using the Canonical Ensemble.
- c) What is the probability $p(n)$ that n particles will be found in the excited state ψ_1 ?
- d) Find the partition function $Z_d(N, T)$ that would apply if the N particles were distinguishable but possessed the same single particle states as above.

Problem 3: Spin Polarization

Consider a 3-dimensional non-interacting quantum gas of $s = 1/2$ Fermions. The two possible spin states are $m_s = 1/2$ and $m_s = -1/2$. In a uniform magnetic field $H\hat{z}$ the single particle energies depend on the direction of the spin relative to the field: $\epsilon = \hbar^2 k^2 / 2m - 2\mu_0 H m_s$ where μ_0 is a constant with the units of magnetic moment.

- Find the density of states as a function of energy separately for the particles with spin parallel and antiparallel to the external field, $D_{1/2}(\epsilon)$ and $D_{-1/2}(\epsilon)$. Make a careful sketch of the total density of states as a function of energy $D(\epsilon) = D_{1/2}(\epsilon) + D_{-1/2}(\epsilon)$.
- For N particles in a volume V at absolute zero ($T = 0$), find an expression for the minimum magnetic field H_0 that will give rise to total polarization of the spins, that is no particles left with the high energy spin orientation.
- Evaluate H_0 in Tesla (one Tesla = 10^4 Gauss) for the electrons in copper where the electronic magnetic moment $\mu_0 = -9.27 \times 10^{-21}$ ergs-gauss $^{-1}$, the particle mass $m = 9.11 \times 10^{-28}$ grams and the number density of conduction electrons is $n \equiv N/V = 8.45 \times 10^{22}$ cm $^{-3}$.
- Evaluate H_0 in Tesla for liquid ^3He where the magnetic moment is that of the nucleus with $\mu_0 = 1.075 \times 10^{-23}$ ergs-gauss $^{-1}$, $m = 5.01 \times 10^{-24}$ grams and the number density of atoms in the liquid is $n = 1.64 \times 10^{22}$ cm $^{-3}$.

One can now buy commercial superconducting solenoids for laboratory research that go up to about 15 Tesla, and at MIT's Francis Bitter National Magnet Laboratory one can obtain continuous fields up to 37 Tesla and pulsed fields as high as 68 Tesla. As you can see, we are a long way from being able to completely polarize either of these systems by brute force.

Problem 4: Relativistic Electron Gas

Consider a 3 dimensional non-interacting quantum gas of ultra-relativistic electrons. In this limit the single particle energies are given by $\epsilon = c\hbar|\vec{k}|$. The density of allowed wavevectors in k space is $V/(2\pi)^3$.

- Find an expression for the Fermi Energy ϵ_F as a function of c , \hbar , N and V .
- Find the density of states as a function of energy $D(\epsilon)$. Sketch the result.

- c) Find the total kinetic energy E of the gas at absolute zero as a function of N and ϵ_F .
- d) Find the pressure exerted by the gas at $T = 0$. How does it depend on the particle density N/V ? Is this a stronger or weaker dependence on density than in the non-relativistic gas?
- e) Assume that this gas represents the electrons in a white dwarf star composed of α particles (a bound state of two neutrons and two protons: a ${}^4\text{He}$ nucleus) and electrons. Express the kinetic energy E_K in terms of the total mass of the star M and its radius R (as well as a handful of physical constants including the α particle mass). Recall that the potential energy of a self gravitating star of radius R with uniform density is given by $E_P = -\frac{3}{5}GM^2/R$ where G is the gravitational constant. Proceed as we did in class to find the equilibrium R as a function of M by minimizing the total energy, $E_T = E_K + E_P$. What can you conclude about the stability of the star in this model?
- f) We know from observation that white dwarfs of mass less than a certain critical mass (of the order of the Sun's mass) are stable. The results of e) show that determining this critical mass will require a more sophisticated model of the star, certainly taking into account the dependence of the mass density on depth in the star and using a dispersion relation for the electrons, $\epsilon(k)$, valid for all energies. However the calculation in e) allows us to find how the critical mass might depend on the important parameters in the problem: c , \hbar , G , and a reference mass such as m_α . Use your results to find an expression for the critical mass of a white dwarf, M_c , in terms of these parameters but neglecting any purely numerical constants of the order of one.