MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2004

Solutions to Problem Set #12

Problem 1: Two Identical Particles

a)

Fermions:	$\|1,1,0>$	ϵ_2	$T=0$ state
	$\|1,0,1>$	ϵ_3	
	$\|0,1,1>$	$\epsilon_2+\epsilon_3$	

b)

Bosons:	$\|2,0,0>$	0	$T=0$ state
	$\|1,1,0>$	ϵ_2	
	$\|1,0,1>$	ϵ_3	
	$\|0,2,0>$	$2\epsilon_2$	
	$\|0,1,1>$	$\epsilon_2+\epsilon_3$	
	$\|0,0,2>$	$2\epsilon_3$	

c) Let $\beta \equiv 1/kT$.

$$Z_F(T) = e^{-\epsilon_2\beta} + e^{-\epsilon_3\beta} + e^{-(\epsilon_2+\epsilon_3)\beta}$$

$$Z_B(T) = 1 + e^{-\epsilon_2\beta} + e^{-\epsilon_3\beta} + e^{-2\epsilon_2\beta} + e^{-(\epsilon_2+\epsilon_3)\beta} + e^{-2\epsilon_3\beta}$$

d)

$$Z_F(T) \approx e^{-\epsilon_2\beta} + e^{-\epsilon_3\beta}$$

$$\begin{aligned} < E >_F &= -\frac{1}{Z_F}\frac{\partial Z_F}{\partial \beta} \\ &\approx \frac{\epsilon_2 e^{-\epsilon_2\beta} + \epsilon_3 e^{-\epsilon_3\beta}}{e^{-\epsilon_2\beta} + e^{-\epsilon_3\beta}} = \frac{\epsilon_2 + \epsilon_3 e^{-(\epsilon_3-\epsilon_2)\beta}}{1 + e^{-(\epsilon_3-\epsilon_2)\beta}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\left[\epsilon_2 + \epsilon_3 e^{-(\epsilon_3-\epsilon_2)\beta}\right]\left[1 - e^{-(\epsilon_3-\epsilon_2)\beta}\right]}{1 - \left(e^{-(\epsilon_3-\epsilon_2)\beta}\right)^2} \\
&= \frac{\epsilon_2 + (\epsilon_2 - \epsilon_3)e^{-(\epsilon_3-\epsilon_2)\beta} - \epsilon_3 e^{-2(\epsilon_3-\epsilon_2)\beta}}{1 - e^{-2(\epsilon_3-\epsilon_2)\beta}} \\
&\approx \underline{\epsilon_2 + (\epsilon_3 - \epsilon_2)e^{-(\epsilon_3-\epsilon_2)\beta}}
\end{aligned}
$$

This result shows a finite $< E >$ at $T = 0$ and energy gap behavior with $\Delta = \epsilon_3 - \epsilon_2$.

$$
\begin{aligned}
< E >_B &= -\frac{1}{Z_B}\frac{\partial Z_B}{\partial \beta} \\
&\approx \frac{\epsilon_2 e^{-\epsilon_2 \beta}}{1 + e^{-\epsilon_2 \beta}} \\
&\approx \underline{\epsilon_2 e^{-\epsilon_2 \beta}}
\end{aligned}
$$

This result shows $< E >= 0$ at $T = 0$ and energy gap behavior with $\Delta = \epsilon_2$.

Problem 2: A Number of Two-State Particles

The two single-particle states available are indicated below.

$$
\begin{array}{lcl}
\psi_1 & \underline{\qquad\qquad} & \epsilon = \Delta \\
 & & \\
\psi_0 & \underline{\qquad\qquad} & \epsilon = 0
\end{array}
$$

a) Use the number of particles in the upper single particle state as the index for the many-particle states:

$$
|n_0, n_1 >= |N - n_1, n_1 > \qquad \text{and} \qquad \underline{E_{n_1} = n_1 \Delta}
$$

$$
n_1 = 0, 1, 2, \cdots N \quad \Rightarrow \underline{N+1} \text{ many-particle states}
$$

b) Let $\beta \equiv 1/kT$.

$$
Z(N, T) = \sum_{n_1=0}^{N} e^{-n_1 \Delta \beta} = \sum_{n_1=0}^{N} \left(e^{-\Delta\beta}\right)^{n_1}
$$

$$
\begin{aligned}
&= \sum_{n_1=0}^{\infty} \left(e^{-\Delta\beta}\right)^{n_1} - \sum_{n_1=N+1}^{\infty} \left(e^{-\Delta\beta}\right)^{n_1} \\
&= \sum_{n_1=0}^{\infty} \left(e^{-\Delta\beta}\right)^{n_1} - \left(e^{-\Delta\beta}\right)^{N+1} \sum_{m=0}^{\infty} \left(e^{-\Delta\beta}\right)^{m} \\
&= \frac{1}{1-e^{-\Delta\beta}} - \frac{e^{-(N+1)\Delta\beta}}{1-e^{-\Delta\beta}} \\
&= \underline{\frac{1-e^{-(N+1)\Delta\beta}}{1-e^{-\Delta\beta}}}
\end{aligned}
$$

c)

$$
p(n_1) = \frac{1}{Z} e^{-n_1\Delta/kT} = \underline{\frac{1-e^{-\Delta/kT}}{1-e^{-(N+1)\Delta/kT}}\, e^{-n_1\Delta/kT}}
$$

d) If the particles are distinguishable, similar, and non-interacting, then

$$
Z_d(N,T) = (Z_{\text{one particle}})^N = \left(1 + e^{-\Delta/kT}\right)^N
$$

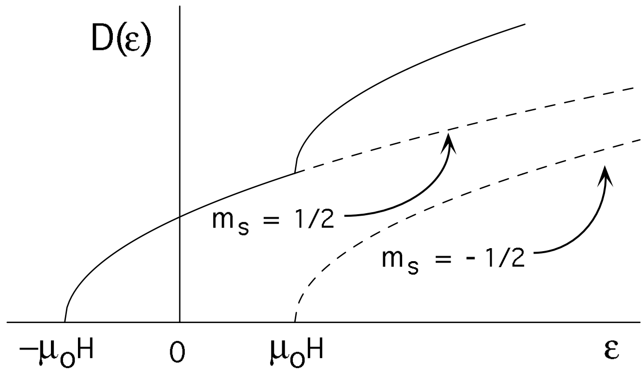
Problem 3: Spin Polarization

a) Recall that in zero field the density of states for spin-$\frac{1}{2}$ Fermions is

$$
\begin{aligned}
D_0(\epsilon) &= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{1/2} \qquad && \epsilon > 0 \\
&= 0 && \epsilon < 0
\end{aligned}
$$

These are equally divided between spin up and spin down. The application of a field shifts all $m_s = \frac{1}{2}$ states down in energy by $\mu_0 H$ and all $m_s = -\frac{1}{2}$ states up by $\mu_0 H$ (assuming a positive μ_0, that is, a magnetic moment parallel rather than anti-parallel to $\vec{S}$). Thus

$$
\begin{aligned}
D_{\frac{1}{2}}(\epsilon) &= \frac{1}{2} D_0(\epsilon + \mu_0 H) \\
D_{-\frac{1}{2}}(\epsilon) &= \frac{1}{2} D_0(\epsilon - \mu_0 H) \\
D(\epsilon) &= \frac{1}{2}[D_0(\epsilon + \mu_0 H) + D_0(\epsilon - \mu_0 H)]
\end{aligned}
$$



b) The filling of $D(\epsilon)$ at $T = 0$ must stop just short of $\epsilon = \mu_0 H$ where the $m_s = -\frac{1}{2}$ states would begin to fill.

$$
\begin{aligned}
N &= \int_{-\mu_0 H}^{\mu_0 H} D(\epsilon)\, d\epsilon = \int_0^{2\mu_0 H} \tfrac{1}{2} \underbrace{D_0(\epsilon)}_{a\epsilon^{1/2}}\, d\epsilon \\
&= \frac{a}{2}\,\frac{2}{3}\,(2\mu_0 H)^{3/2} \\
&= \frac{1}{3}\left[\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\right](2\mu_0 H)^{3/2} \\
6\pi^2 N/V &= \left(\frac{4m\mu_0 H}{\hbar^2}\right)^{3/2} \\
(6\pi^2 N/V)^{2/3} &= \frac{4m\mu_0}{\hbar^2}\, H \\
H_0 &= \underline{\frac{1}{\mu_0}\,\frac{\hbar^2 (6\pi^2 N/V)^{2/3}}{4m}}
\end{aligned}
$$

c) Using cgs units

$$H_0 = 4.22 \times 10^{-54} \frac{1}{\mu_0 M} (N/V)^{2/3}$$

Using $M = 9.11 \times 10^{-28}$g, $\mu_0 = -9.27 \times 10^{-21}$ergs-gauss^{-1}, and $n = 8.45 \times 10^{22}$cm^{-3} gives

$$H_0 = 9.6 \times 10^8 \text{gauss} = \underline{9.6 \times 10^4 \text{Tesla}}.$$

The negative μ_0 means that the electron spins are polarized anti-parallel to the direction of $\vec{H}$.

d) For ^{3}He, $M = 5.01 \times 10^{-24}$g, $\mu_0 = 1.075 \times 10^{-23}$ergs-gauss^{-1}, and $n = 1.64 \times 10^{22}$cm^{-3}. So in this system

$$H_0 = 5.1 \times 10^7 \text{gauss} = \underline{5.1 \times 10^3 \text{Tesla}}.$$

Problem 4: Relativistic Electron Gas

a)

$$\begin{aligned}
D_{\text{wavevectors}}(k) &= \frac{V}{(2\pi)^3} \qquad D_{\text{states}}(k) = \frac{2V}{(2\pi)^3} \\
N &= \left(\frac{4}{3}\pi k_f^3\right)\frac{2V}{(2\pi)^3} \quad \Rightarrow \quad k_f = (3\pi^2 N/V)^{1/3} \\
\epsilon &= c\hbar|\vec{k}| \quad \Rightarrow \quad \underline{\epsilon_f = c\hbar(3\pi^2 N/V)^{1/3}}
\end{aligned}$$

b)

$$\begin{aligned}
\#(\epsilon) &= \left(\frac{4}{3}\pi \underbrace{k^3(\epsilon)}_{(\epsilon/c\hbar)^3}\right)\left(\frac{2V}{(2\pi)^3}\right) = \frac{1}{3\pi^2}V\left(\frac{1}{c\hbar}\right)^3 \epsilon^3 \\
D(\epsilon) &= \frac{d\#}{d\epsilon} = \underline{\frac{V}{\pi^2}\left(\frac{1}{c\hbar}\right)^3 \epsilon^2}
\end{aligned}$$

c)

$$\begin{aligned}
D(\epsilon) &= a\epsilon^2 \\
N &= \int_0^{\epsilon_\text{f}} D(\epsilon)\, d\epsilon = \int_0^{\epsilon_\text{f}} a\epsilon^2\, d\epsilon = \tfrac{1}{3}a\epsilon_\text{f}^3 \\
E &= \int_0^{\epsilon_\text{f}} \epsilon D(\epsilon)\, d\epsilon = \int_0^{\epsilon_\text{f}} a\epsilon^3\, d\epsilon = \tfrac{1}{4}a\epsilon_\text{f}^4 \\
&= \underline{\tfrac{3}{4}N\epsilon_\text{f}}
\end{aligned}$$

d)

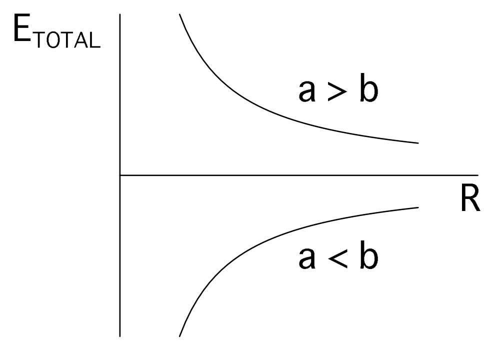
$$P = -\underbrace{\left(\frac{\partial E}{\partial V}\right)_{N,S}}_{dS=0 \text{ at } T=0} = -\frac{3}{4}N\left(\frac{\partial \epsilon_\text{f}}{\partial V}\right)_N = \underline{\frac{1}{4}(N/V)\epsilon_\text{f} \propto (N/V)^{4/3}}$$

This pressure rises less steeply with density, $(N/V)^{4/3}$, than is the case for the non-relativistic gas, $(N/V)^{5/3}$.

e) For a white dwarf composed of α particles and electrons,

$$\begin{aligned}
V &= \frac{4}{3}\pi R^3 \\
M &\approx N_\alpha m_\alpha = \frac{1}{2}N_e m_\alpha \Rightarrow N_e = 2(M/m_\alpha) \\
E_K &= \frac{3}{4}N_e \epsilon_F = \frac{3}{4}N_e c\hbar(3\pi^2 N_e/V)^{1/3} \\
&= \frac{3}{2}c\hbar(\frac{M}{m_\alpha})\left(\frac{9\pi}{2}\frac{M}{m_\alpha}\frac{1}{R^3}\right)^{1/3} \\
&= \underline{\frac{3}{2}\left(\frac{9\pi}{2}\right)^{1/3} c\hbar\left(\frac{M}{m_\alpha}\right)^{4/3}\frac{1}{R}}
\end{aligned}$$

The R dependence of the two contributions to the total energy is straight forward: $E_{\mathrm{K}} = a/R$ and $E_{\mathrm{P}} = -b/R$ where a and b are known expressions. Then $E_{\mathrm{TOTAL}} = (a-b)/R$ which is never stable. The condition $a = b$ is a special case, a dividing line between collapse and infinite expansion.



f) The condition $a = b$ probably gives a combination of parameters indicating the dimensions of a critical mass which would emerge when a more detailed calculation is done. What mass gives $a \sim b$? Neglect numerical factors of the order of one.

$$\begin{aligned}
c\hbar\left(\frac{M}{m_\alpha}\right)^{4/3} &\sim GM^2 \\
\frac{c\hbar}{Gm_\alpha^{4/3}} &\sim M^{2/3} \\
M &\sim \underline{\left(\frac{c\hbar}{Gm_\alpha^2}\right)^{3/2} m_\alpha}
\end{aligned}$$

The Chandrasekhar limit for the maximum possible mass of a white dwarf is

$$M_{\mathrm{Ch}} = 0.20 \left(\frac{Z}{A}\right) \left(\frac{ch}{Gm_p^2}\right)^{3/2} m_p$$

where Z/A is the average ratio of atomic number to atomic weight of the stellar constituents. Note that it has the same form as our expression. For $Z/A = 0.5$ (α particles) this gives $M_{\mathrm{Ch}} = 1.4 M_{\mathrm{Sun}}$.