

## Wavefunctions, One Particle

Hamiltonian  $\hat{\mathcal{H}}(\hat{\vec{r}}, \hat{\vec{p}}, \hat{\vec{s}})$

Wavefunction  $\psi_n(\vec{r}, \vec{s})$

$\vec{r}$  and  $\vec{s}$  are the variables.

$n$  is a state index and could have several parts.

For an  $e^-$  in hydrogen  $\psi = \psi_{n,l,m_l,m_s}(\vec{r}, \vec{s})$

$$\hat{\mathcal{H}}(\hat{\vec{r}}, \hat{\vec{p}}, \hat{\vec{s}}) \psi_n(\vec{r}, \vec{s}) = E_n \psi_n(\vec{r}, \vec{s})$$

$\psi_n(\vec{r}, \vec{s})$  often factors into space and spin parts.

$$\psi_n(\vec{r}, \vec{s}) = \psi_{n'}^{\text{space}}(\vec{r}) \psi_{n''}^{\text{spin}}(\vec{s})$$

$$\psi_n^{\text{space}}(x) \propto e^{-\alpha x^2/2} H_n(\sqrt{\alpha} x) \quad \text{H.O. in 1 dimension}$$

$$\psi_n^{\text{space}}(\vec{r}) \propto e^{i\vec{k}\cdot\vec{r}} \quad \text{free particle in 3 dimensions}$$

$$\psi_{n''}^{\text{spin}}(\vec{s})$$

Spin is an angular momentum so for a given value of the magnitude  $S$  there are  $2S + 1$  values of  $m_S$ .

For the case of  $S = 1/2$  the eigenfunctions of the  $z$  component of  $\vec{s}$  are  $\phi_{1/2}(\vec{s})$  and  $\phi_{-1/2}(\vec{s})$

$$\hat{S}_z \phi_{1/2}(\vec{s}) = \frac{\hbar}{2} \phi_{1/2}(\vec{s})$$

$$\hat{S}_z \phi_{-1/2}(\vec{s}) = -\frac{\hbar}{2} \phi_{-1/2}(\vec{s})$$

$\psi_{n''}^{\text{spin}}(\vec{s})$  is not necessarily an eigenfunction of  $\hat{S}_z$ . For example one might have

$$\psi_{n''}^{\text{spin}}(\vec{s}) = \frac{1}{\sqrt{2}} \phi_{1/2}(\vec{s}) + \frac{1}{\sqrt{2}} \phi_{-1/2}(\vec{s})$$

In some cases  $\psi_n(\vec{r}, \vec{s})$  may not factor into space and spin parts. For example one may find

$$\psi_n(x, \vec{s}) = f(x) \phi_{1/2}(\vec{s}) + g(x) \phi_{-1/2}(\vec{s})$$

Many Distinguishable Particles, Same Potential,  
No Interaction

Lump space and spin variables together

$\vec{r}_1, \vec{s}_1 \rightarrow 1 \quad \vec{r}_2, \vec{s}_2 \rightarrow 2$  etc.

$$\hat{\mathcal{H}}(1, 2, \dots, N) = \hat{\mathcal{H}}_0(1) + \hat{\mathcal{H}}_0(2) + \dots + \hat{\mathcal{H}}_0(N)$$

In this expression the single particle Hamiltonians all have the same functional form but each has arguments for a different particle.

The same set of single particle energy eigenstates is available to every particle, but each may be in a different one of them. The energy eigenfunctions of the system can be represented as products of the single particle energy eigenfunctions.

$$\psi_{\{n\}}(1, 2, \dots, N) = \psi_{n_1}(1)\psi_{n_2}(2) \cdots \psi_{n_N}(N)$$

$\{n\} \equiv \{n_1, n_2, \dots, n_N\}$ . There are  $N$  #s, but each  $n_i$  could have an infinite range.

$$\hat{\mathcal{H}}(1, 2, \dots, N) \psi_{\{n\}}(1, 2, \dots, N) = E_{\{n\}} \psi_{\{n\}}(1, 2, \dots, N)$$

Many Distinguishable Particles, Same Potential,  
Pairwise Interaction

$$\hat{\mathcal{H}}(1, 2, \dots, N) = \sum_{i=1}^N \hat{\mathcal{H}}_0(i) + \frac{1}{2} \sum_{i \neq j} \hat{\mathcal{H}}_{\text{int}}(i, j)$$

The  $\psi_{\{n\}}(1, 2, \dots, N)$  are no longer energy eigenfunctions; however, they could form a very useful basis set for the expansion of the true energy eigenfunctions.

## Indistinguishable Particles

$$\hat{P}_{ij} f(\cdots i \cdots j \cdots) \equiv f(\cdots j \cdots i \cdots)$$

$$(\hat{P}_{ij})^2 = \hat{I} \quad \Rightarrow \quad \text{eigenvalues of } \hat{P}_{ij} \text{ are } +1, -1$$

It is possible to construct many-particle wavefunctions which are symmetric or anti-symmetric under this interchange of two particles.

$$\hat{P}_{ij} \psi^{(+)} = \psi^{(+)}$$

$$\hat{P}_{ij} \psi^{(-)} = -\psi^{(-)}$$

Identical  $\Rightarrow$  no physical operation distinguishes between particle  $i$  and particle  $j$ . Mathematically, this means that for all physical operators  $\hat{O}$

$$[\hat{O}, \hat{P}_{ij}] = 0$$

$\Rightarrow$  eigenfunctions of  $\hat{O}$  must also be eigenfunctions of  $\hat{P}_{ij}$ .

$\Rightarrow$  energy eigenfunctions  $\psi_E$  must be either  $\psi_E^{(+)}$  or  $\psi_E^{(-)}$ .

⇒ states differing only by the interchange of the spatial and spin coordinates of two particles are the same state.

Relativistic quantum mechanics requires

$$\begin{array}{lll} \text{integer spin} & \leftrightarrow & \psi_E^{(+)} \quad [\text{Bosons}] \\ \text{half-integer spin} & \leftrightarrow & \psi_E^{(-)} \quad [\text{Fermions}] \end{array}$$