

**Solutions to Problem Set #10**

**Problem 1:** The Big Bang

If the expansion is adiabatic,  $\Delta S = 0$ .

$$\begin{aligned} S &= - \left( \frac{\partial F}{\partial T} \right)_V \\ &= - \frac{\partial}{\partial T} \left( - \frac{1}{45} \frac{\pi^2}{c^3 \hbar^3} (kT)^4 V \right) \\ &= \frac{4}{45} \frac{\pi^2}{c^3 \hbar^3} k^4 T^3 V \end{aligned}$$

From this result we see that the product  $T^3 V$  remains constant during the expansion. Therefore

$$\frac{V(3\text{K})}{V(3000\text{K})} = \frac{3000^3}{3^3} = 10^9$$

**Problem 2:** Comet Hale-Bopp

a) The total power radiated by the sun

$$\begin{aligned} &= \text{surface area of the sun} \times e(T_{\text{sun}}) \\ &= 4\pi R_{\text{sun}}^2 e(T_{\text{sun}}) \end{aligned}$$

The fraction intercepted by the comet

$$\begin{aligned} &= \frac{\text{cross-sectional area of the comet}}{4\pi r^2} \\ &= \frac{\pi R_{\text{comet}}^2}{4\pi (200R_{\text{sun}})^2} \end{aligned}$$

The power absorbed by the comet

$$\begin{aligned} &= 4\pi R_{\text{sun}}^2 e(T_{\text{sun}}) \times \frac{\pi R_{\text{comet}}^2}{4\pi (200R_{\text{sun}})^2} \\ &= \frac{\pi R_{\text{comet}}^2 e(T_{\text{sun}})}{(200)^2} \end{aligned}$$

The power radiated by the comet

$$\begin{aligned} &= \text{surface area of the comet} \times e(T_{\text{comet}}) \\ &= 4\pi R_{\text{comet}}^2 e(T_{\text{comet}}) \end{aligned}$$

In the steady state, the power absorbed equals the power radiated:

$$\frac{\pi R_{\text{comet}}^2 e(T_{\text{sun}})}{(200)^2} = 4\pi R_{\text{comet}}^2 e(T_{\text{comet}})$$

Note that the radius of the comet cancels out here.

$$\frac{\sigma T_{\text{sun}}^4}{(200)^2} = 4\sigma T_{\text{comet}}^4$$

The Stefan-Boltzmann constant cancels out here. Finally

$$T_{\text{comet}} = T_{\text{sun}} \left( \frac{1}{16 \times 10^4} \right)^{1/4} = \frac{T_{\text{sun}}}{20} = \frac{6000}{20} = \underline{300\text{K}}$$

b) The coma surrounding the nucleus could absorb or reflect away part of the incoming radiation from the sun. Also, the nucleus might not be a “black” absorber. Both effects would lead to a colder nucleus. On the other hand, the coma would reflect some radiation emitted from the comet back toward it (a greenhouse effect) and if the nucleus is a poorer absorber, it is also a poorer emitter. A quantitative estimate of the temperature of the nucleus requires more information and detailed heat transport calculations.

c) 300K is close to the earth’s average surface temperature. It is not a coincidence. The temperature we calculate is independent of the radius of the body and inversely proportional to the square of the body’s distance from the sun. The sun does heat the planets:

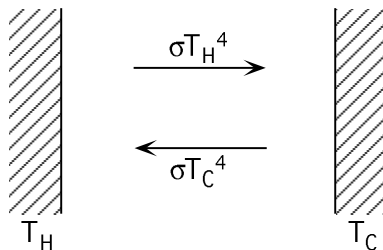
- Planets get colder as one goes farther from the sun,
- The seasons on the earth depend on the continent’s average inclination toward the sun.

But, there is also heat coming from the nuclear fission of radioactive elements in the earth’s interior:

- The earth's surface was once much hotter (molten) when this heat source (as well as a gravitational component) was much greater than the heat flux from the sun.
- The earth's temperature increases with depth below the surface (this is evident in deep shaft mines) which implies a heat flow toward the surface from the interior.

**Problem 3: Super Insulation**

a)



$$J = \sigma T_H^4 - \sigma T_C^4 = \underline{\sigma(T_H^4 - T_C^4)} \equiv J_0$$

b) Since the sheet is neither cooling nor warming, the flux in from  $T_H$  must equal the flux out to  $T_C$ .

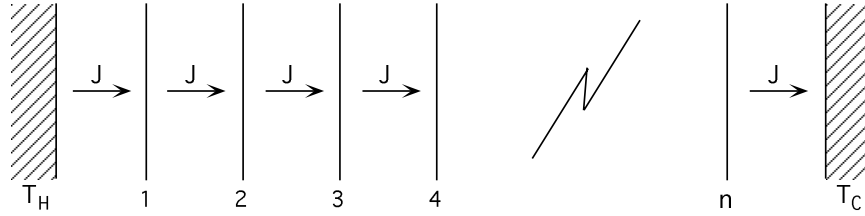
$$\sigma(T_H^4 - T^4) = \sigma(T^4 - T_C^4) \Rightarrow T_H^4 + T_C^4 = 2T^4$$

$$\underline{T = \left( \frac{T_H^4 + T_C^4}{2} \right)^{1/4}}$$

c)

$$\begin{aligned} J &= \sigma(T_H^4 - T^4) \\ &= \sigma\left[T_H^4 - \frac{1}{2}(T_H^4 + T_C^4)\right] \\ &= \underline{\frac{1}{2}\sigma(T_H^4 - T_C^4) = \frac{1}{2}J_0} \end{aligned}$$

d) The net energy gain for any one sheet is zero, since the temperatures are constant in the steady state. This implies that the energy flux,  $J$ , between any two adjacent sheets is the same.



$$\begin{aligned}
 T_{i-1}^4 - T_i^4 &= J/\sigma \Rightarrow T_i^4 = T_{i-1}^4 - J/\sigma \\
 T_1^4 &= T_H^4 - J/\sigma \\
 T_2^4 &= T_1^4 - J/\sigma = T_H^4 - 2J/\sigma \\
 T_3^4 &= T_2^4 - J/\sigma = T_H^4 - 3J/\sigma \\
 &\dots\dots\dots \\
 T_C^4 &= T_n^4 - J/\sigma = T_H^4 - (n+1)J/\sigma
 \end{aligned}$$

Solving the final equation for  $J$  gives the result

$$J = \frac{\sigma}{n+1} [T_H^4 - T_C^4] = \frac{1}{n+1} J_0$$

**Problem 4:** Properties of Blackbody Radiation

a) In class we found the following expression for the internal energy of the thermal radiation field:

$$U(T, V) = \frac{1}{15} \frac{\pi^2}{c^3 \hbar^3} (kT)^4 V;$$

therefore

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V = \frac{4}{15} \pi^2 \left( \frac{kT}{\hbar c} \right)^3 kV.$$

Since  $(kT/\hbar c)$  has the units of  $(\text{length})^{-1}$ ,  $C_V$  has the units of  $k$ , which is correct.

$$\frac{k}{\hbar c} = \frac{1.38 \times 10^{-16} \text{ ergs K}^{-1}}{1.054 \times 10^{-27} \text{ erg-s} \times 300 \times 10^{10} \text{ cm s}^{-1}} = 4.36 \text{ K}^{-1} \text{ cm}^{-1}$$

For  $1 \text{ m}^3 = 10^6 \text{ cm}^3$  and  $T = 300 \text{ K}$ ,

$$C_V = \frac{4}{15} \pi^2 (4.36 \times 300)^3 \times 10^6 k = \underline{5.89 \times 10^{15} k = 0.813 \text{ ergs/K}}$$

which is not much.

b)  $C_V = \frac{3}{2}Nk$  for a monatomic gas. To get the same  $C_V$  as above,

$$\begin{aligned}\frac{3}{2}N &= 5.89 \times 10^{15} \\ N &= 3.93 \times 10^{15} \\ N/V &= 3.93 \times 10^{15}/10^6 = \underline{3.9 \times 10^9 \text{ atoms/cm}^3}\end{aligned}$$

This is ten orders of magnitude less than the STP density of  $2.69 \times 10^{19}$  atoms/cm<sup>3</sup>.

c) The energy stored in an electric field is

$$U_E = \frac{1}{8\pi} \int \vec{E} \cdot \vec{E} dV \quad \Rightarrow \quad \langle U_E \rangle = \frac{1}{8\pi} \int \langle \vec{E} \cdot \vec{E} \rangle dV$$

For thermal radiation  $\langle \vec{E} \cdot \vec{E} \rangle$  is independent of position, so

$$\langle U_E \rangle = \frac{1}{8\pi} \int \langle \vec{E} \cdot \vec{E} \rangle dV \quad \Rightarrow \quad \langle U_E \rangle / V = \frac{1}{8\pi} \langle \vec{E} \cdot \vec{E} \rangle$$

where  $U_E$  is in ergs and  $E$  is in statvolts/cm (Gaussian CGS units). In a radiation field the energy stored in the magnetic field  $\langle U_B \rangle$  is equal to that in the electric field, so

$$\langle U_B \rangle = \langle U_E \rangle = \frac{1}{2}U(T, V) \quad \Rightarrow \quad \underline{\underline{U(T, V)/V = \frac{1}{4\pi} \langle E^2 \rangle}}$$

Now use the result from a)

$$\begin{aligned}U(T, V)/V &\equiv u(T) = \frac{1}{15}\pi^2 \left(\frac{kT}{c\hbar}\right)^4 c\hbar \\ &= \frac{\pi^2}{15}(4.36T)^4 c\hbar \\ &= 7.52 \times 10^{-15}T^4 \text{ ergs/cm}^3\end{aligned}$$

$$\langle E^2 \rangle = 4\pi \times 7.52 \times 10^{-15}T^4 = 7.65 \times 10^{-4} \text{ (statvolts/cm)}^2$$

$$\langle E^2 \rangle^{1/2} = 2.77 \times 10^{-2} \text{ statvolts/cm} = \underline{\underline{8 \text{ volts/cm}}}$$

**Problem 5:** Radiation Pressure

From class we have the relation  $P = \frac{1}{3}u(T)$  for the thermal radiation field. Using the expression for  $U(T, V) = u(T)V$  from problem 4a we have

$$P_{\text{radiation}} = \frac{1}{3} \frac{\pi^2}{15} \left( \frac{kT}{c\hbar} \right)^3 kT = \frac{\pi^2}{45} (4.36T)^3 kT$$

For a classical gas  $P_{\text{kinetic}} = nkT$ . Setting  $P_{\text{radiation}} = P_{\text{kinetic}}$  gives

$$n = \frac{\pi^2}{45} (4.36T)^3$$

Solving for  $T$  and using a STP density of  $n = 2.69 \times 10^{19}$  leads to

$$T = \left[ \frac{2.69 \times 10^{19} \times 45}{\pi^2 (4.36)^3} \right]^{1/3} = \underline{1.1 \times 10^6 \text{K}}$$