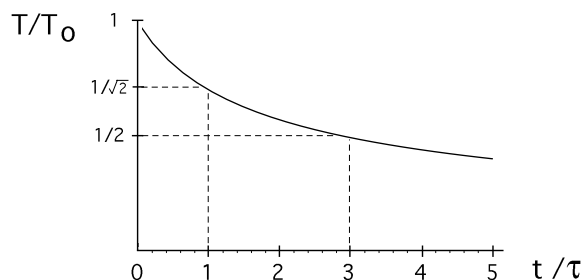


**Exam #4**

**Problem 1** (35 points) Cooling of a White Dwarf Star



Just after a white dwarf star is formed it begins a long slow radiational cooling process which will eventually reduce it to a cold dark ember. In this problem you will find its temperature as a function of time. You may assume that

- There are no longer any heat sources in the star,
- The thermal conductivity is so high that the temperature  $T$  is essentially uniform throughout the star,
- The heat capacity is that of the nearly degenerate ( $kT \ll \epsilon_f$ ) electron gas and has the form  $C_V = \gamma VT^n$  where  $V$  is the volume and  $\gamma$  is a constant,
- The surface of the star is a perfect absorber of radiation at all frequencies.

- a) What is the value of the exponent  $n$  in the expression for the heat capacity?
- b) Find an expression for the derivative of the total energy of the star with respect to temperature,  $dE/dT$ .
- c) Find an expression for the derivative of the total energy of the star with respect to time,  $dE/dt$ .
- d) Find the differential equation which determines the time evolution of the temperature. Give your result in terms of  $\gamma$ , the radius of the star  $R$ , and any physical constants which you think necessary. Check to see that the equation is consistent with your common sense expectation for  $T(t)$ .

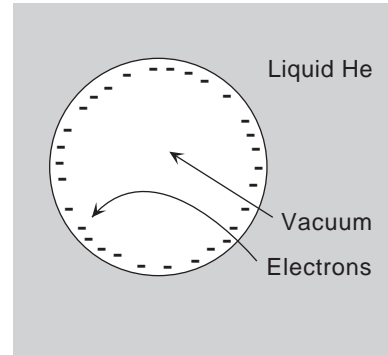
Hint: The correct differential equation is not hard to solve. By solving it you can check your result against the behavior plotted in the figure.

**Problem 2** (35 points) Two-Dimensional Electron Gas

Consider a gas of non-interacting, spin 1/2 electrons confined to move in two dimensions. For a rectangular sample with dimensions  $L_x$  and  $L_y$ , the wavevectors allowed by periodic boundary conditions are  $\vec{k} = (2\pi/L_x)m\hat{x} + (2\pi/L_y)n\hat{y}$  where  $m$  and  $n$  can take on all positive and negative integer values.

- Find  $D(\vec{k})$ , the density of allowed wavevectors as a function of  $\vec{k}$ .
- Find  $D(\epsilon)$ , the density of single particle states as a function of their energy  $\epsilon$ . Make a carefully labeled sketch of your result.

Researchers have proposed a novel system for studying the properties of a highly degenerate two-dimensional electron gas. The electrons can be accumulated on the inside surface of a spherical bubble in liquid helium, as shown in the figure.



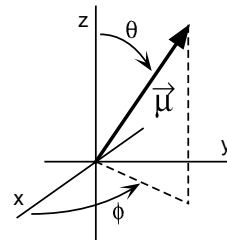
The equilibrium radius of the bubble is found by minimizing its energy  $E(R)$  with respect to its radius  $R$ .

$$E(R) = E_{\text{Coulomb}} + E_{\text{Surface Tension}} + E_{pdV} + E_{\text{Electron Gas}}$$

- Find at  $T = 0$  the contribution to this sum from the kinetic energy of the two-dimensional electron gas,  $E_{\text{Electron Gas}}$ , as a function of  $R$  and  $N$ , the number of electrons in the bubble.
- Find at  $T = 0$  the total magnetic moment of the bubble,  $M(H)$ , as a function of the applied magnetic field  $H$ . Make a carefully labeled sketch of  $M(H)$  for all positive values of the field. [Recall that the magnetic moment of a single electron is quantized,  $\mu_z = \pm\mu_0$ , and its contribution to the energy is  $-\mu_z H_z$ ]

**Problem 3** (30 points) Classical Paramagnet

Consider a collection of  $N$  non-interacting classical magnetic moments. Each moment  $\vec{\mu}$  has a fixed magnitude  $\mu_0$  and its orientation is specified by the two angles  $0 \leq \theta < \pi$  and  $0 \leq \phi < 2\pi$ . In the presence of a magnetic field of magnitude  $H$  pointing in the  $z$  direction, the energy of an individual moment is given by  $\epsilon = -\vec{\mu} \cdot \vec{H} = -\mu_0 H \cos(\theta)$ .



Applying the canonical ensemble one finds that

$$p(\theta, \phi) = Z_1^{-1} \exp\left(\frac{\mu_0 H}{kT} \cos(\theta)\right) \quad \text{where} \quad \int p(\theta, \phi) \sin(\theta) d\theta d\phi = 1.$$

The partition function for a single moment  $Z_1$  can be expressed in terms of the dimensionless combination of parameters  $\eta = \mu_0 H/kT$ :

$$Z_1(\eta) = \int_0^{2\pi} d\phi \int_0^\pi e^{\eta \cos \theta} \sin(\theta) d\theta = \frac{4\pi}{\eta} \sinh(\eta).$$

- a) Using the information given above, find an expression for the magnetic moment of the sample,  $M = N \langle \mu_z \rangle$ , in terms of an appropriate derivative of the single moment partition function.
- b) Find an analytic expression for  $M$  in terms of  $\mu_0$ ,  $N$ , and  $\eta$ .
- c) Find expressions for the temperature dependence of  $M$  in both the high temperature and low temperature regimes. Use these results to make a careful sketch of  $M$  as a function of  $T$  for a fixed value of  $H$ .
- d) Two concepts used to explain the properties of the *quantum* paramagnet were Curie Law behavior and energy gap behavior.
  - i) Does the classical paramagnet exhibit Curie Law behavior? Explain.
  - ii) Does the classical paramagnet exhibit energy gap behavior? Explain.
- e) In the classical paramagnet, as in the quantum paramagnet, the link between statistical mechanics and thermodynamics is provided by the Gibbs free energy:  $-kT \ln Z = G(T, H)$ . Find an expression for the entropy of the system in terms of  $Z_1(\eta)$  and its derivatives. You do not have to carry out any derivatives of  $Z_1(\eta)$  that might be involved.

## Work in simple systems

Hydrostatic system	$-PdV$
Surface film	$\$dA$
Linear system	$\mathcal{F}dL$
Dielectric material	$\mathcal{E}d\mathcal{P}$
Magnetic material	$HdM$

Thermodynamic Potentials when work done on the system is  $dW = Xdx$

Energy	$E$	$dE = TdS + Xdx$
Helmholtz free energy	$F = E - TS$	$dF = -SdT + Xdx$
Gibbs free energy	$G = E - TS - Xx$	$dG = -SdT - xdx$
Enthalpy	$H = E - Xx$	$dH = TdS - xdx$

## Results from hyperbolic trigonometry

$$\begin{aligned} \sinh(u) &= (e^u - e^{-u})/2 & \cosh(u) &= (e^u + e^{-u})/2 \\ \tanh(u) &= \sinh(u)/\cosh(u) & \coth(u) &= 1/\tanh(u) \\ \frac{d}{dx}(\sinh u) &= (\cosh u) \frac{du}{dx} & \frac{d}{dx}(\cosh u) &= (\sinh u) \frac{du}{dx} \end{aligned}$$

Limiting behavior of

	as $u \rightarrow 0$	as $u \rightarrow \infty$
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$\sinh(u)$	$u$	$e^u/2$
$\cosh(u)$	$1 + u^2/2$	$e^u/2$
$\tanh(u)$	$u$	$1$
$\coth(u)$	$1/u + \frac{1}{3}u$	$1$

## Radiation laws

Kirchoff's law:  $e(\omega, T)/\alpha(\omega, T) = \frac{1}{4}cu(\omega, T)$  for all materials where  $e(\omega, T)$  is the emissive power and  $\alpha(\omega, T)$  the absorptivity of the material and  $u(\omega, T)$  is the universal blackbody energy density function.

Stefan-Boltzmann law:  $e(T) = \sigma T^4$  for a blackbody where  $e(T)$  is the emissive power integrated over all frequencies. ( $\sigma = 56.9 \times 10^{-9}$  watt-m<sup>-2</sup>K<sup>-4</sup>)