

**Exam #3**

**Problem 1** (30 points) Weakly Interacting Bose Gas

At low temperatures the entropy and isothermal compressibility of a weakly interacting Bose gas can be approximated by

$$S(T, V) = \frac{5}{2} a T^{3/2} V$$

$$\kappa_T \equiv -\frac{1}{V} \left. \frac{\partial V}{\partial P} \right|_T = \frac{1}{2c} V^2$$

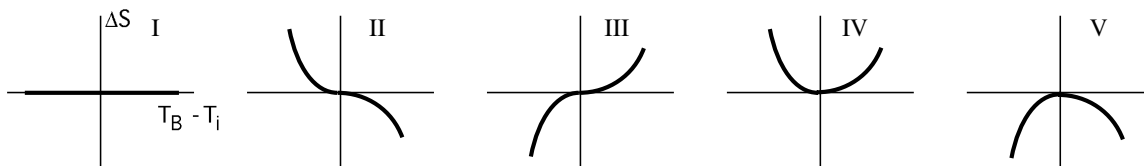
where  $a$  and  $c$  are constants. In the limit of low temperature and high volume the pressure  $P$  and the internal energy density  $E/V$  approach zero.

- a) (13) Find the equation of state  $P(T, V)$ .
- b) (13) Find the internal energy  $E(T, V)$ .
- c) (4) Does this model for the gas obey the third law of thermodynamics? Explain the reasoning behind your answer.

**Problem 2** (10 points) Entropy Changes

A solid of fixed volume, initially at a temperature  $T_i$ , is suddenly immersed in a bath of fluid at temperature  $T_B$ . Eventually the solid comes to thermal equilibrium with the bath at  $T_f$ . No work is done during this process and the solid plus bath are isolated from the rest of the world.

Which of the following graphs could represent the behavior of the total entropy change  $\Delta S$  (solid plus bath) as a function of the temperature difference  $T_B - T_i$ ? Explain your choice.



**Problem 3** (30 points) Dust Grains in Space

Astronomers have discovered that there exist in the interstellar medium clouds of “dust grains” whose chemical composition may include silicates (like sand) or carbon-containing compounds (like graphite or silicon carbide). Evidence indicates that many of these grains have a needle like shape.

If the rotational motion of a dust grain is to be described in terms of its principal axes (1,2, and 3) the appropriate coordinates (angles) and canonically conjugate angular momenta are  $\theta_1, \theta_2, \theta_3$  and  $L_1, L_2, L_3$ . In terms of these variables the classical Hamiltonian for a single grain is given by

$$\mathcal{H} = \frac{1}{2I_1}L_1^2 + \frac{1}{2I_2}L_2^2 + \frac{1}{2I_3}L_3^2$$

where  $I_i$  is the moment of inertia about the  $i^{\text{th}}$  principal axis. Assume that a dust cloud is in thermal equilibrium at a temperature  $T$ .

- a) (10) Find an analytic expression (no unevaluated integrals) for the joint probability density function for the canonical variables,  $p(\theta_1, \theta_2, \theta_3, L_1, L_2, L_3)$ , for a single dust grain.
- b) (7) Assume that axis 3 is parallel to the long axis of the grain and that  $I_3 \ll I_1 = I_2$ . Will the angular momentum of a dust grain be more likely to be parallel or perpendicular to the long (3) axis? Find a quantitative result to support your contention.
- c) (7) Find the rotational contribution of the grains to the entropy of a dust cloud containing  $N$  dust grains.
- d) (3) Does this model for the rotational motion of the dust grains obey the third law of thermodynamics? Explain the reasoning behind your answer.
- e) (3) Should this model of rotational motion exhibit energy gap behavior? Why?

**Problem 4** (30 points) Adsorption On a Stepped Surface

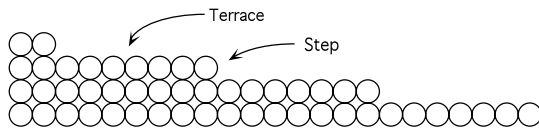


Fig.1 Cut through crystal perpendicular to the surface

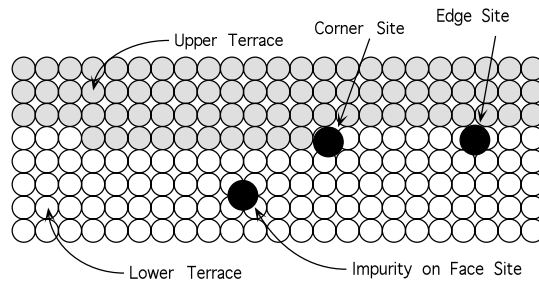


Fig.2 Looking down on the surface, showing adsorption sites

If a perfect crystal is cleaved along a symmetry direction, the resulting surface could expose a single geometrically flat plane of atoms. Alternatively, if the crystal is cut at a slight angle with respect to this direction, the resulting surface might take the form of a series of terraces of fixed width separated by steps of height corresponding to one atomic layer. This situation is illustrated in figure 1. The steps themselves may not be straight lines, but may have kinks when viewed from above as shown in figure 2. If impurity atoms were adsorbed on such a surface, their energy could depend on where they reside relative to the steps.

Consider  $N$  identical xenon atoms adsorbed on a silicon surface which has a total of  $M$  possible adsorption sites. There are three different types of site that the xenon could occupy. They would prefer to snuggle into a corner site at a kink in a step. One percent of the sites are corner sites, and their energy defines the zero of the energy scale for adsorbed atoms. Next in preference are edge sites. Fourteen percent of the  $M$  sites are edge sites with an energy  $\Delta$  above that of a corner site. The majority (85%) of the adsorption sites are face sites, but they have an energy which is  $1.5\Delta$  above that of the corner sites.  $M$  is so large compared to  $N$  competition for a given site can be neglected; the xenon atoms can be considered completely independent. You may neglect the kinetic energy of the adsorbed atoms.

- a) (7) Find the partition function,  $Z(N, T)$ , for the xenon atoms in terms of the parameters  $M$  and  $\Delta$ .
- b) (4) Find the ratio of xenon atoms on face sites to the number on corner sites.
- c) (10) Find an expression for the heat capacity of the adsorbed xenon atoms in the limit  $kT \ll \Delta$ .
- d) (3) What is the probability that a given xenon atom will be found on a face site in the limit  $kT \gg \Delta$ ?
- e) (3) What is the limit of the entropy of this system as  $T \rightarrow \infty$ ?
- f) (3) Should this model for adsorbed atoms exhibit energy gap behavior? Why?

## PARTIAL DERIVATIVE RELATIONSHIPS

Let  $x, y, z$  be quantities satisfying a functional relation  $f(x, y, z) = 0$ . Let  $w$  be a function of any two of  $x, y, z$ . Then

$$\left(\frac{\partial x}{\partial y}\right)_w \left(\frac{\partial y}{\partial z}\right)_w = \left(\frac{\partial x}{\partial z}\right)_w$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{\left(\frac{\partial y}{\partial x}\right)_z}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

## INTEGRALS

$$\int_0^{\infty} \frac{x^n}{a} e^{-x/a} dx = n! a^n$$

$$\int_{-\infty}^{\infty} \frac{x^{2n}}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} dx = 1 \cdot 3 \cdot 5 \cdots (2n-1)\sigma^{2n}$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

## STERLING'S APPROXIMATION

$$\ln K! \approx K \ln K - K \quad \text{when } K \gg 1$$

## SOLID ANGLE INCREMENT

$$d\Omega = \sin \theta d\theta d\phi$$

## WORK IN SIMPLE SYSTEMS

System	Intensive quantity	Extensive quantity	Work
Hydrostatic system	$P$	$V$	$-PdV$
Wire	$\mathcal{F}$	$L$	$\mathcal{F}dL$
Surface	$\mathcal{S}$	$A$	$\mathcal{S}dA$
Reversible cell	$E$	$Z$	$E dZ$
Dielectric material	$\mathcal{E}$	$\mathcal{P}$	$\mathcal{E}d\mathcal{P}$
Magnetic material	$H$	$M$	$HdM$

## THERMODYNAMIC POTENTIALS

For a system in which the increment of work done on the system is  $dW = Xdx$

Energy	$E$	$dE = TdS + Xdx$
Helmholtz free energy	$F = E - TS$	$dF = -SdT + Xdx$
Gibbs free energy	$G = E - TS - Xx$	$dG = -SdT - xdx$
Enthalpy	$H = E - Xx$	$dH = TdS - xdx$