MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.044 Statistical Physics I — Spring Term 2004

Solutions to Problem Set #6

Problem 1: Classical Magnetic Moments

a)

$$\begin{aligned} -\mu \leq & \quad m_i & \leq \mu \\ -\mu N \leq & \quad M & \leq \mu N \\ -N(\mu H) \leq & \quad E & \leq N(\mu H) \end{aligned}$$

b) There are $2N$ microscopic variables necessary to specify the state of the system. Some possibilities include the x and z component of each spin, the z component and the angle in the $x\,y$ plane for each spin, or the polar angles θ and ϕ for each spin.

c)

$$\frac{H}{T} = -\left(\frac{\partial S}{\partial M}\right)_N = -k\,\frac{1}{\Omega}\left(\frac{\partial \Omega}{\partial M}\right)_N \qquad \text{using } S = -k\ln\Omega$$

$$= -k\,\frac{1}{\Omega}\left(-\frac{2M}{(2/3)\mu^2 N}\right)\Omega = k\left(\frac{M}{(1/3)N\mu^2}\right)$$

$$\Rightarrow \underline{M(H,T) = \left(\frac{N\mu^2}{3k}\right)\frac{H}{T}} \qquad \text{the Curie law result}$$

d) The expression found in c) allows M to grow without bound as $T \to 0$. But $|M|$ is bounded by μN. Thus the expression can only be trusted as an approximation when

$$\begin{aligned} M(H,T) &<< \mu N \\ \frac{N\mu^2}{3kT} &<< \mu N \\ &\Rightarrow \underline{kT >> (1/3)\mu H} \end{aligned}$$

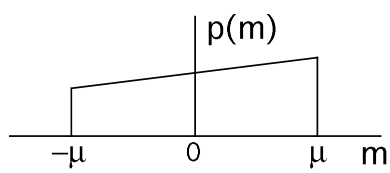
This result says that the "thermal energy", kT, must be much greater than the maximum energy allowable for a single spin.

e) To find Ω' reduce N by 1 and reduce M by m.

$$\underline{\Omega' = (2\mu)^{N-1}\exp\left[-\frac{(M-m)^2}{(2/3)(N-1)\mu^2}\right]}$$

f)

$$\begin{aligned} p(m) &= \frac{\Omega'}{\Omega} = \frac{(2\mu)^{N-1}}{(2\mu)^N} \, \frac{\exp\left[-\frac{(M-m)^2}{(2/3)(N-1)\mu^2}\right]}{\exp\left[-\frac{M^2}{(2/3)N\mu^2}\right]} \\ &= \frac{1}{2\mu} \, \exp\left[-\frac{M^2-2Mm+m^2}{(2/3)(N-1)\mu^2}\right] \exp\left[\frac{M^2}{(2/3)N\mu^2}\right] \\ &\approx \frac{1}{2\mu} \, \exp \underbrace{\left[\frac{3mM}{N\mu^2}\right]}_{\text{small since } M<<\mu N} \\ &\approx \underline{\frac{1}{2\mu} \left(1 + (\frac{3M}{N\mu^2})\, m\right) \qquad -\mu \le m \le \mu} \end{aligned}$$



Now check the normalization.

$$\begin{aligned} \int_{-\mu}^{\mu} p(m)\, dm &= \int_{-\mu}^{\mu} \frac{1}{2\mu}\, dm + \int_{-\mu}^{\mu} \frac{3M}{2N\mu^3}\, m\, dm \\ &= 1 + 0 = 1 \end{aligned}$$

g)

$$\begin{aligned} < m > &= \int p(m)\, m\, dm \\ &= \underbrace{\int_{-\mu}^{\mu} \frac{1}{2\mu}\, m\, dm}_{=0} + \int_{-\mu}^{\mu} \frac{3M}{2N\mu^3}\, m^2\, dm \\ &= \frac{3M}{2N\mu^3} \left[\frac{1}{3} m^3 \right]_{-\mu}^{\mu} = \underline{\frac{M}{N}} \end{aligned}$$

This tells us that the total magnetization M is N times the average moment of an individual dipole, the result that one would expect on physical grounds.

Problem 2: A Strange Chain

a) The number of ways of choosing n_+ elements from a total of N is $N!/(N-n_+)!n_+!$. It follows that

$$
\begin{aligned}
\Omega(N, n_+) &= \frac{N!}{(N-n_+)!n_+!} \\
S(N, n_+) &= k \ln \Omega \\
&\approx k\{N \ln N - (N-n_+) \ln(N-n_+) - n_+ \ln n_+ \underbrace{-N + (N-n_+) + n_+}_{=0}\} \\
&= \underline{k\{N \ln N - (N-n_+) \ln(N-n_+) - n_+ \ln n_+\}}
\end{aligned}
$$

b)

$$
\begin{aligned}
\frac{\mathcal{F}}{T} &= -\left(\frac{\partial S}{\partial L}\right)_{N,E} \\
&= -\frac{\partial S}{\partial n_+} \underbrace{\frac{\partial n_+}{\partial L}}_{1/2l} \\
&= -\frac{k}{2l}\left\{\frac{N-n_+}{N-n_+} + \ln(N-n_+) - \frac{n_+}{n_+} - \ln n_+\right\} \\
-\frac{2l\mathcal{F}}{kT} &= \ln\left(\frac{N-n_+}{n_+}\right) \\
\mathcal{F}(N, T, n_+) &= \underline{-\frac{kT}{2l} \ln\left(\frac{N-n_+}{n_+}\right)}
\end{aligned}
$$

c) Now rearrange the last result, take the exponential of both sides and solve for n_+.

$$
\begin{aligned}
\exp[-2l\mathcal{F}/kT] &= \frac{N-n_+}{n_+} \\
n_+ &= N \frac{1}{1+\exp[-2l\mathcal{F}/kT]}
\end{aligned}
$$

Next, use the expression for n_+ to find L.

$$
\begin{aligned}
L &= l(2n_+ - N) = Nl\left(\frac{2}{1+\exp[]} - \frac{1+\exp[]}{1+\exp[]}\right) \\
&= Nl\,\frac{1-\exp[-2l\mathcal{F}/kT]}{1+\exp[-2l\mathcal{F}/kT]} \\
&= Nl\,\frac{\exp[l\mathcal{F}/kT]-\exp[-l\mathcal{F}/kT]}{\exp[l\mathcal{F}/kT]+\exp[-l\mathcal{F}/kT]} \\
&= \underline{Nl\tanh(l\mathcal{F}/kT)}
\end{aligned}
$$

For high temperatures, where $kT >> l\mathcal{F}$, $\tanh x \to x$ for small x, so

$$L \approx \left(\frac{Nl^2}{kT}\right)\mathcal{F}.$$

The fact that the length L is proportional to the tension $\mathcal{F}$ shows that Hooke's law applies to this system, at least for high temperatures.

d)

$$
\begin{aligned}
\alpha &\equiv L^{-1}\left(\frac{\partial L}{\partial T}\right)_{\mathcal{F}} \\
&= \frac{1}{L}\left(-\frac{1}{T}(L)\right) \\
&= \underline{-\frac{1}{T}}
\end{aligned}
$$

Problem 3: Classical Harmonic Oscillators

a) In x space

$$E = \sum_{i=1}^{2N} x_i^2$$

is a sphere in $2N$ dimensions with radius $\sqrt{E}$. Its volume is $\pi^N E^N/N!$. The corresponding volume in pq space is

$$
\begin{aligned}
\Omega(E,N) &= \frac{\pi^N}{N!}(\sqrt{2m}\,)^N\left(\sqrt{\frac{2}{m\omega^2}}\right)^N E^N \\
&= \underline{\left(\frac{2\pi}{\omega}\right)^N \frac{1}{N!}\,E^N}
\end{aligned}
$$

b)

$$S(E,N) = k \ln \Omega(E,N) = \underline{k \ln \left\{ \left(\frac{2\pi}{\omega} \right)^N \frac{1}{N!} E^N \right\}}$$

c)

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_N = k \frac{N E^{-1} \{\}}{\{\}} = \frac{Nk}{E}$$

$$\Rightarrow \underline{E = NkT} \qquad \underline{C_N = Nk}$$

d) Let Ω' be the volume in a phase space for $N-1$ oscillators of total energy $E-\epsilon$ where $\epsilon = (1/2m)p_i^2 + (m\omega^2/2)q_i^2$. Since the oscillators are all similar, $< \epsilon >= E/N = kT$.

$$p(p_i, q_i) = \Omega'/\Omega$$

$$\Omega' = \left(\frac{2\pi}{\omega} \right)^{N-1} \frac{1}{(N-1)!} (E-\epsilon)^{N-1}$$

$$\frac{\Omega'}{\Omega} = \left(\frac{2\pi}{\omega} \right)^{-1} \frac{N!}{(N-1)!} \left(\frac{E-\epsilon}{E} \right)^N \frac{1}{E-\epsilon}$$

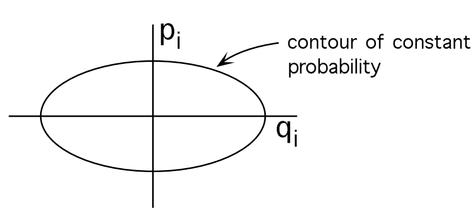
$$= \frac{\omega}{2\pi} \underbrace{\frac{N}{E-\epsilon}}_{\approx <\epsilon>^{-1}} \underbrace{\left(1 - \frac{\epsilon}{E} \right)^N}_{\approx \exp[-\epsilon/<\epsilon>]}$$

$$p(p_i, q_i) = \frac{1}{(2\pi/\omega) < \epsilon >} \exp[-\epsilon/ < \epsilon >]$$

$$= \frac{1}{(2\pi/\omega)kT} \exp[-p_i^2/2mkT] \exp[-(m\omega^2/2kT)q_i^2]$$

$$= \underline{\left(\frac{1}{\sqrt{2\pi mkT}} \exp[-p_i^2/2mkT] \right) \left(\frac{1}{\sqrt{2\pi(kT/m\omega^2)}} \exp[-q_i^2/2(kT/m\omega^2)] \right)}$$

$$= p(p_i) \times p(q_i) \Rightarrow p_i \text{ and } q_i \text{ are S.I.}$$



Problem 4: Quantum Harmonic Oscillators

a)

$$E = (1/2)\hbar\omega N + \hbar\omega M$$

$$\Omega(E,N) = \underline{\Omega(M,N) = \frac{(M+N-1)!}{M!(N-1)!}}$$

b)

$$S(M,N) = k \ln \Omega(M,N)$$

$$\approx \underline{k\{(M+N-1)\ln(M+N-1) - M\ln M - (N-1)\ln(N-1)\}}$$

c)

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = \left(\frac{\partial S}{\partial M}\right)_N \left(\frac{\partial M}{\partial E}\right)_N$$

$$= \frac{k}{\hbar\omega}\{1 + \ln(M+N-1) - 1 - \ln M\}$$

$$= \frac{k}{\hbar\omega}\ln\left(\frac{M+N-1}{M}\right)$$

Neglect 1 compared to N.

$$\frac{M+N}{M} = \exp[\hbar\omega/kT]$$

$$M = N(\exp[\hbar\omega/kT] - 1)^{-1}$$

$$E = \underline{N\hbar\omega\left(\frac{1}{2} + \frac{1}{\exp[\hbar\omega/kT] - 1}\right)}$$

$$(\exp[\hbar\omega/kT] - 1)^{-1} \rightarrow \exp[-\hbar\omega/kT] \qquad \text{when } kT << \hbar\omega,$$

$$\rightarrow \frac{kT}{\hbar\omega} - \frac{1}{2} \qquad \text{when } kT >> \hbar\omega.$$

Take careful note of the 1/2 in the last expression. It cancels the other 1/2 appearing in the expression for E in the high temperature limit.

d) Remove one oscillator in state n with energy

$$\epsilon = \hbar\omega(1/2 + n), \quad <\epsilon> = \hbar\omega(1/2 + <n>), \quad <n> = M/N.$$

Then for the reduced phase space $N \to N-1$ and $M \to M-n$.

$$\begin{aligned}
p(n) &= \frac{\Omega'}{\Omega} \\
&= \frac{\Omega(M-n, N-1)}{\Omega(M,N)} \\
&= \frac{(M+N-n-2)!}{(M+N-1)!} \, \frac{M!}{(M-n)!} \, \frac{(N-1)!}{(N-2)!} \\
&= \frac{\overbrace{M(M-1)\cdots(M-n-1)}^{n \text{ terms}}}{\underbrace{(M+N-1)(M+N-2)\cdots(M+N-n-1)}_{n+1 \text{ terms}}} \times (N-1) \\
&\approx \frac{N}{M+N}\left(\frac{M}{M+N}\right)^n \\
&= \underline{\frac{1}{1+<n>}\left(\frac{<n>}{1+<n>}\right)^n}
\end{aligned}$$

Note the the final result for $p(n)$ is properly normalized.