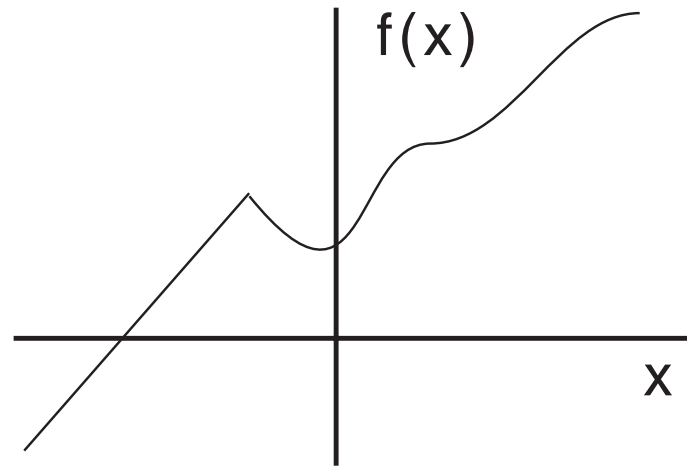


Functions of a random variable

Given: $p_x(\zeta)$ and $f(x)$

Find: $p_f(\eta)$



A. Sketch $f(x)$. Find where $f(x) < \eta$

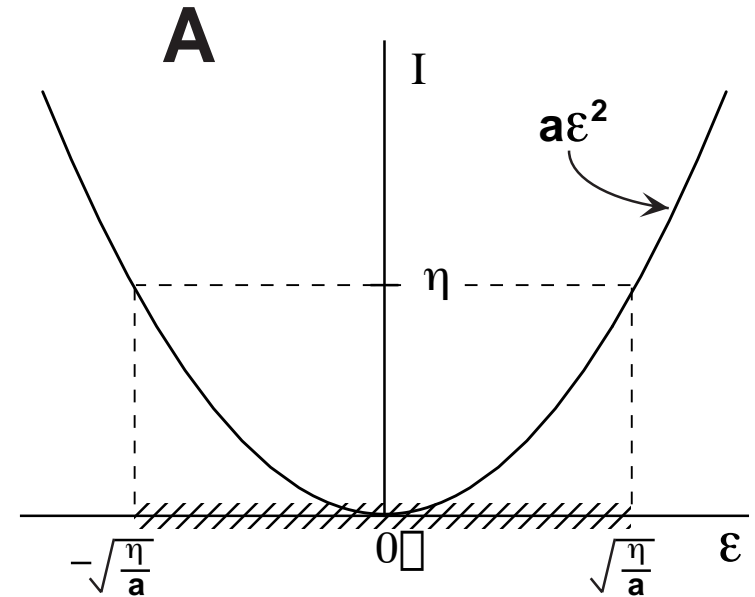
B. Integrate to find $P_f(\eta)$.

C. Differentiate to find $p_f(\eta)$.

Example Intensity of light

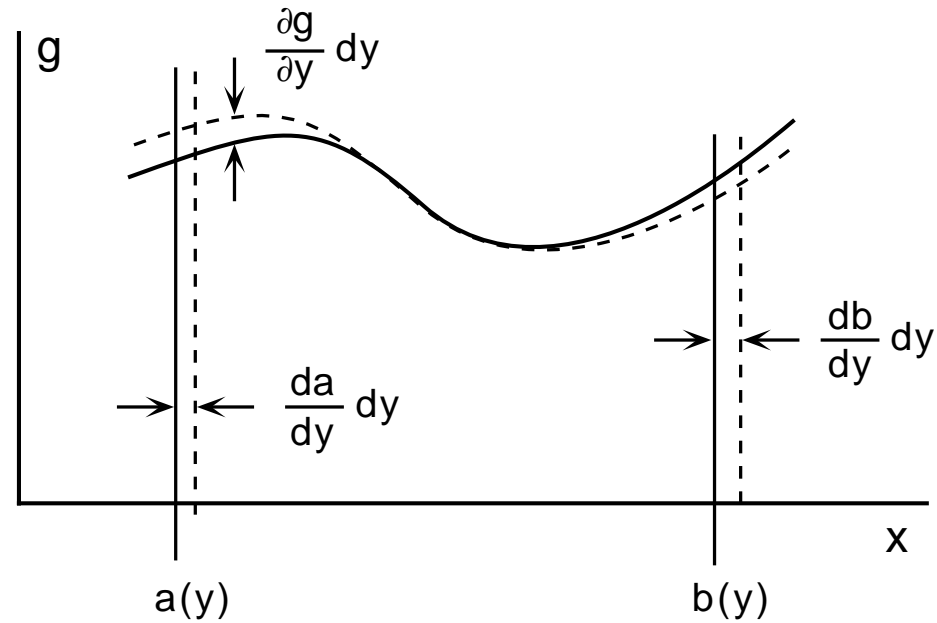
$$I = a\mathcal{E}^2$$

$$p(\mathcal{E}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\mathcal{E}^2/2\sigma^2]$$



B

$$P_I(\eta) = \int_{-\sqrt{\eta/a}}^{\sqrt{\eta/a}} p_{\mathcal{E}}(\zeta) d\zeta$$



$$\frac{d}{dy} \int_{a(y)}^{b(y)} g(y, x) dx =$$

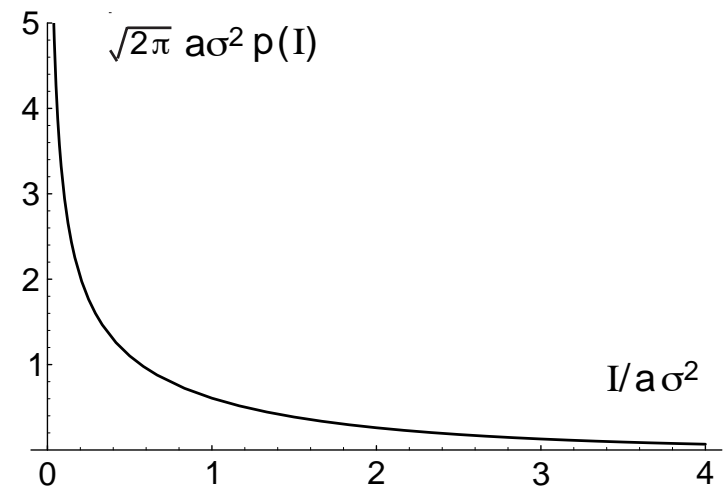
$$g(y, x = b(y)) \frac{db(y)}{dy} - g(y, x = a(y)) \frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial g(y, x)}{\partial y} dx$$

C In general

$$\begin{aligned} p_I(\eta) &= \frac{1}{2} \frac{1}{\sqrt{\eta a}} p_{\mathcal{E}}(\sqrt{\eta/a}) - \left(-\frac{1}{2} \frac{1}{\sqrt{\eta a}}\right) p_{\mathcal{E}}(-\sqrt{\eta/a}) \\ &= \frac{1}{2} \frac{1}{\sqrt{\eta a}} [p_{\mathcal{E}}(\sqrt{\eta/a}) + p_{\mathcal{E}}(-\sqrt{\eta/a})] \end{aligned}$$

In our particular case

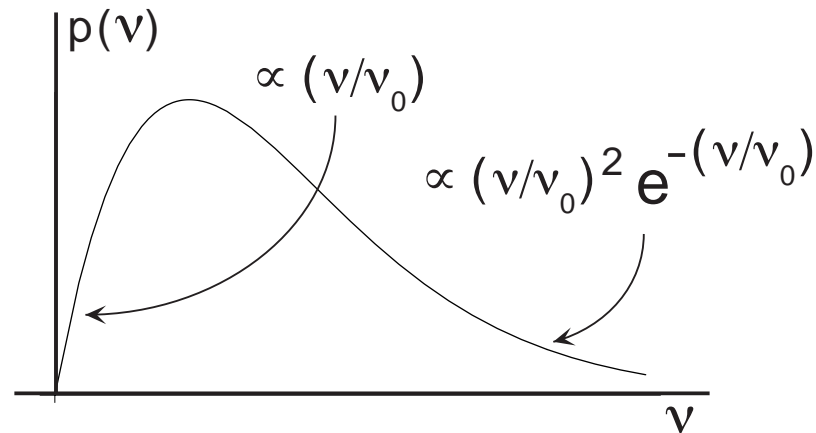
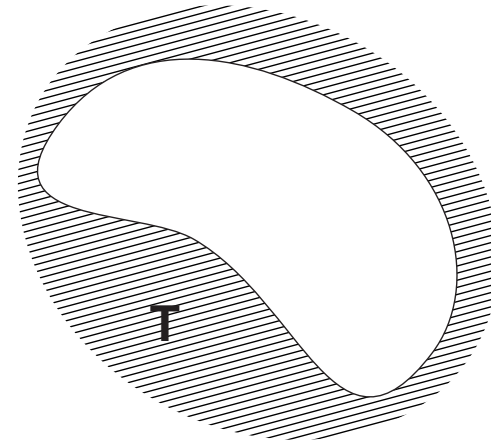
$$p(I) = \frac{1}{\sqrt{2\pi a\sigma^2 I}} \exp\left[-\frac{I}{2a\sigma^2}\right]$$



Example Black Body Radiation

$$p(\nu) = \frac{1}{\underbrace{2\zeta(3)}_{1/2.404}} \frac{1}{\nu_0} \frac{(\nu/\nu_0)^2}{\exp[\nu/\nu_0] - 1}$$

$$\nu_0 = kT/h$$

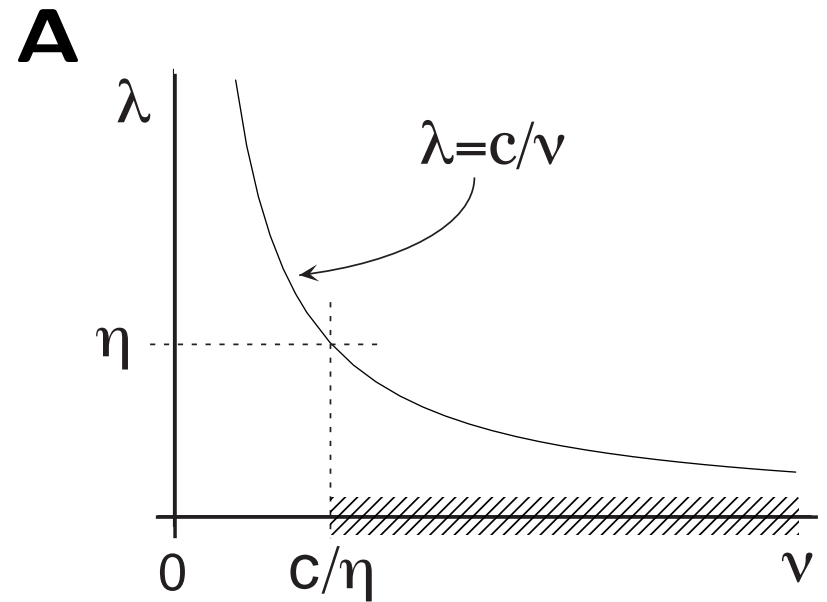


Given $\lambda = c/\nu$ and $p(\nu)$

Find $p(\lambda)$

B

$$P_\lambda(\eta) = \int_{c/\eta}^{\infty} p_\nu(\zeta) d\zeta$$



C

In general

$$p_\lambda(\eta) = -(-c/\eta^2) p_\nu(c/\eta)$$

In our case

$$p_\lambda(\eta) = \frac{c}{\eta^2} \frac{1}{2.404} \frac{1}{\nu_0} \frac{(c/\eta\nu_0)^2}{\exp[(c/\eta\nu_0)] - 1}$$

Let $\lambda_0 \equiv c/\nu_0$, then

$$p_\lambda(\eta) = \frac{1}{2.404} \frac{1}{\lambda_0} \left(\frac{\eta}{\lambda_0}\right)^{-4} \frac{1}{\exp[(\eta/\lambda_0)^{-1}] - 1}$$

$$\text{As } (\lambda/\lambda_0) \rightarrow 0 \quad \frac{1}{\exp[(\lambda/\lambda_0)^{-1}] - 1} \rightarrow e^{-(\lambda/\lambda_0)^{-1}}$$

$$\text{As } (\lambda/\lambda_0) \rightarrow \infty \quad \frac{1}{\exp[(\lambda/\lambda_0)^{-1}] - 1} \rightarrow \frac{1}{(1 + (\lambda/\lambda_0)^{-1} - 1)} \rightarrow (\lambda/\lambda_0)$$

