

## Sums of Random Variables

Consider  $n$  RVs  $x_i$  and let  $s \equiv \sum_{i=1}^n x_i$ .

If the RVs are statistically independent, then

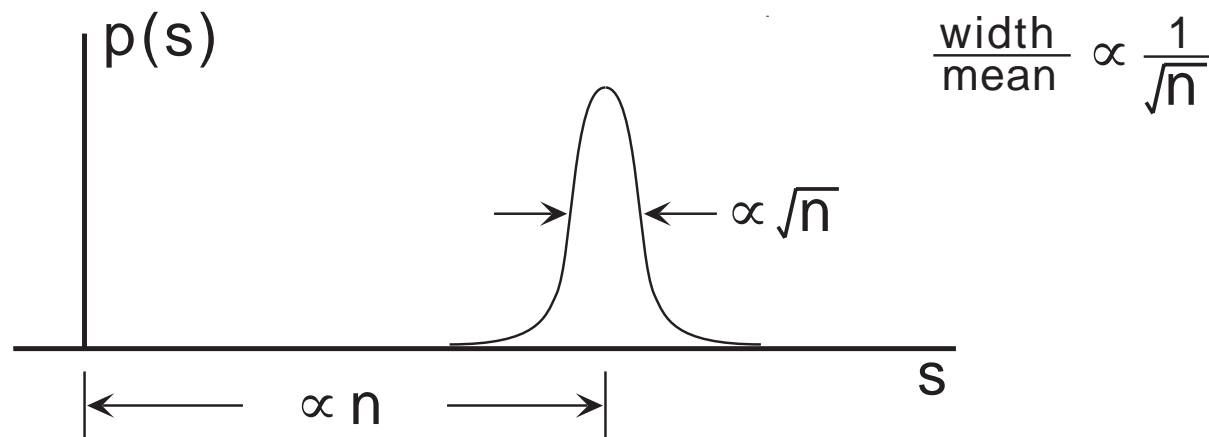
$$\langle s \rangle = \sum_i \langle x_i \rangle$$

$$\text{Var}(s) = \sum_i \text{Var}(x_i)$$

- The individual  $p(x_i)$  could be quite different
- Both continuous and discrete RVs could be present
- True for any  $n$
- Even if one RV dominates the sum

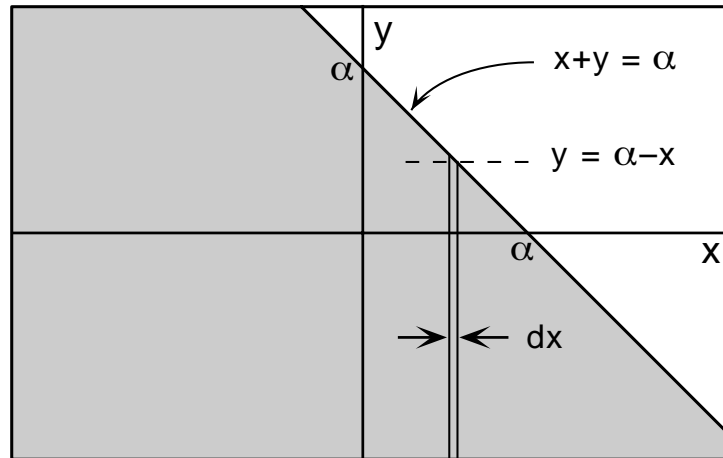
Results have a special meaning when

- 1) The means are finite ( $\neq 0$ )
- 2) The variances are finite ( $\neq \infty$ )
- 3) No subset dominates the sum
- 4)  $n$  is large



Given  $p(x, y)$ , find  $p(s \equiv x + y)$

**A**



**B**

$$P_s(\alpha) = \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\alpha - \zeta} d\eta p_{x,y}(\zeta, \eta)$$

**C** 
$$p_s(\alpha) = \int_{-\infty}^{\infty} d\zeta p_{x,y}(\zeta, \alpha - \zeta)$$

This is a general result;  $x$  and  $y$  need not be S.I.

Application to the Jointly Gaussian RVs in Section 2 shows that  $p(s)$  is a Gaussian with zero mean and a Variance  $= 2\sigma^2(1 + \rho)$ .

In the special case that  $x$  and  $y$  are S.I.

$$p_s(\alpha) = \int_{-\infty}^{\infty} d\zeta p_x(\zeta) p_y(\alpha - \zeta) = \int_{-\infty}^{\infty} d\zeta' p_x(\alpha - \zeta') p_y(\zeta')$$

The mathematical operation is called “convolution”.

$$p \otimes q \equiv \int_{-\infty}^{\infty} p(z)q(x - z)dz = f(x).$$

## Example

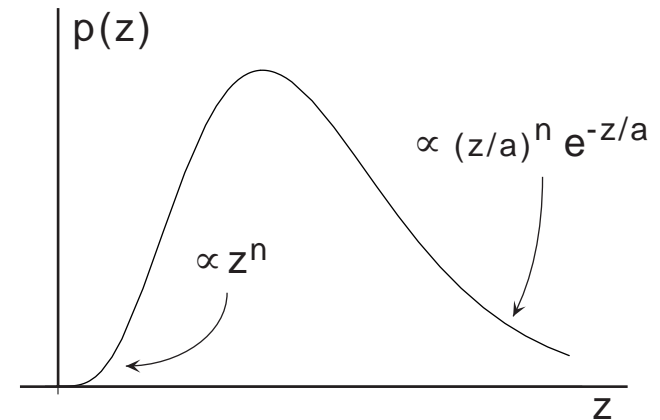
Given:

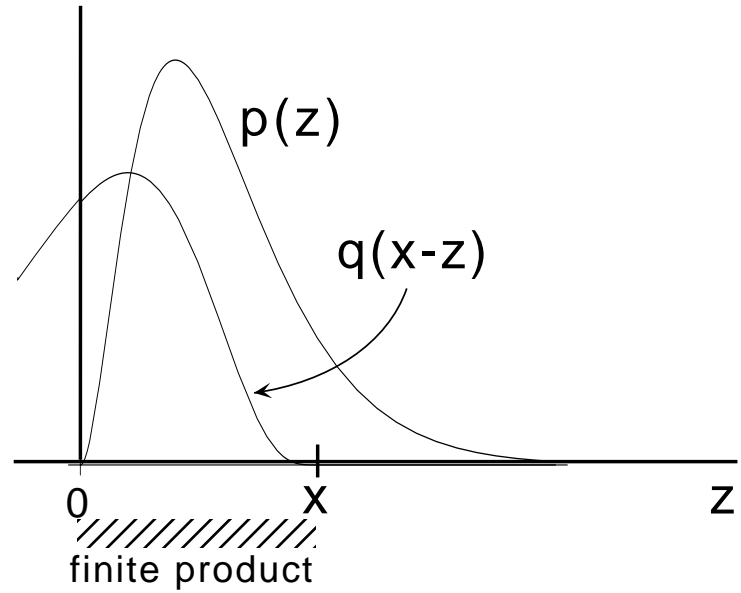
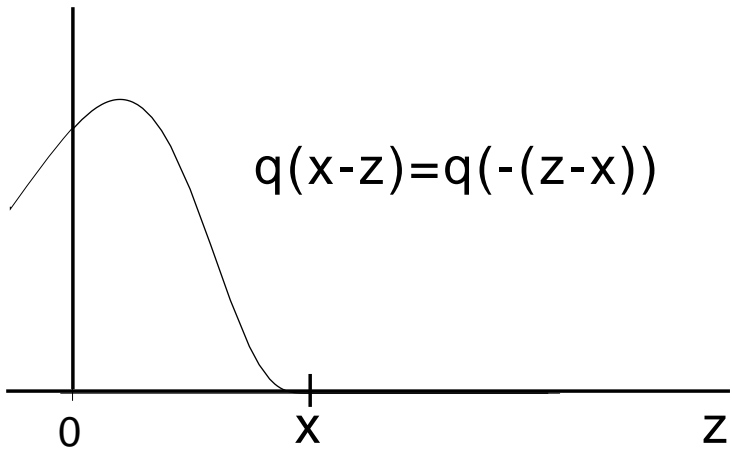
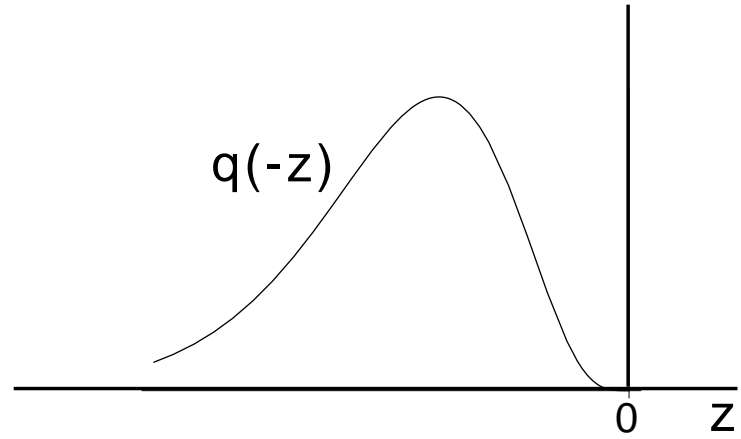
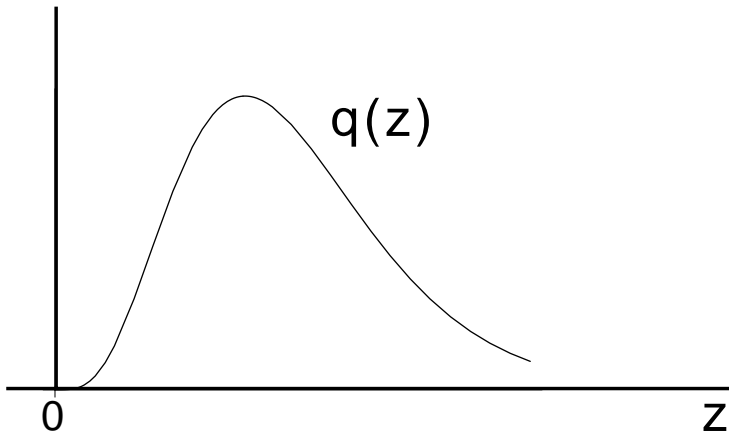
$$p(z) = \frac{1}{n!a} (z/a)^n \exp(-z/a)$$

$$q(z) = \frac{1}{m!a} (z/a)^m \exp(-z/a)$$

$$0 < z \text{ and } n, m = 0, 1, 2, \dots$$

Find:  $p \otimes q$



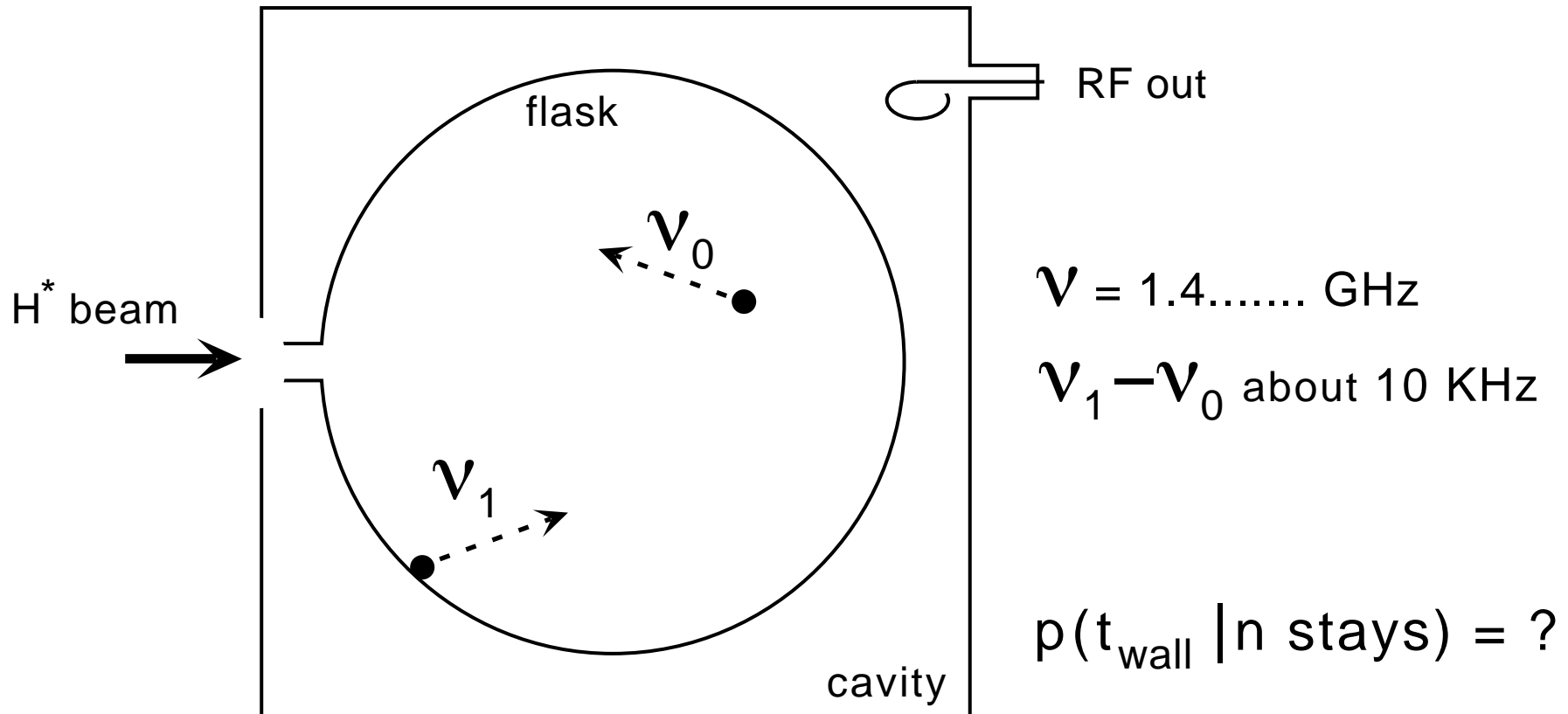


$$\begin{aligned}
p \otimes q &= \frac{1}{n!m!} \frac{1}{a^2} \int_0^x \left(\frac{z}{a}\right)^n \left(\frac{x-z}{a}\right)^m e^{-z/a} e^{-(x-z)/a} dz \\
&= \frac{1}{n!m!} \frac{1}{a} \left(\frac{1}{a}\right)^{n+m+1} e^{-x/a} \int_0^x z^n (x-z)^m dz \\
&= \frac{1}{n!m!} \frac{1}{a} \left(\frac{x}{a}\right)^{n+m+1} e^{-x/a} \underbrace{\int_0^1 \zeta^n (1-\zeta)^m d\zeta}_{\frac{n!m!}{(n+m+1)!}}
\end{aligned}$$

$$p \otimes q = \frac{1}{(n + m + 1)!} \frac{1}{a} \left(\frac{x}{a}\right)^{n+m+1} e^{-x/a}$$

a function of the same class

# Example Atomic Hydrogen Maser



$$t_{\text{wall}} = \sum_{i=1}^n t^i, \quad \text{Each stay is S.I.}$$

$$p(t | 1) = (1/\tau) e^{-t/\tau}$$

$$p(t | 2) = p(t | 1) \otimes p(t | 1) = (1/\tau)(t/\tau) e^{-t/\tau}$$

$$p(t | 3) = p(t | 2) \otimes p(t | 1) = (1/2)(1/\tau)(t/\tau)^2 e^{-t/\tau}$$

$$p(t | n) = \frac{1}{(n-1)!} \frac{1}{\tau} \left(\frac{t}{\tau}\right)^{n-1} e^{-t/\tau}$$

