Quantum Physics II (8.05) Fall 2004
Assignment 5

Massachusetts Institute of Technology
Physics Department
October 7, 2004

Due October 14, 2004
7:00pm

This week we will cover coherent states and the classical limit of quantum mechanics. We will then start our discussion of two state systems. General introduction and examples, spin in a magnetic field, eigenstates and spin precession.

Reading Assignment for week five of the course

- A detailed discussion of two state systems can be found in Cohen-Tannoudji. The first few sections overlap with the material we discussed at the beginning of the course. The Stern-Gerlach experiment is described in §IV.A.1. The theoretical description is given in §IV.A.2. Discussion of sequential measurements of spin as an illustration of basic quantum measurments is in §IV.B. General two state systems are introduced in §IV.C (this has considerable overlap with the posted lecture notes). Complement C_{IV} discusses the "fictitious" spin associated with any two state system. Complement F_{IV} is on nuclear magnetic resonance.

Problem Set 5

1. **Coherent States of the Harmonic Oscillator** [25 points]

   One definition of a coherent state is *an eigenstate of the annihilation operator*:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{1}$$

   Again, we label the state succinctly by its eigenvalue, α.

   (a) Show that

$$|\alpha\rangle = Ne^{\alpha\hat{a}^\dagger}|0\rangle \tag{2}$$

   is an eigenstate of $\hat{a}$ with eigenvalue α. (α is any complex number.)

   (b) Show that the state $|x_0\rangle = e^{-ix_0\hat{p}/\hbar}|0\rangle$ given in equation (4) of the notes on coherent states by R.L. Jaffe, and discussed in lecture, is of the form (2). What is the eigenvalue α?

   (c) Write the state (2) as a superposition of the energy eigenstates $|n\rangle$. Calculate the normalization constant N. What is the probability that a particle in the state $|\alpha\rangle$ has energy E_n?

   (d) Show that any two different coherent states $|\alpha\rangle$ and $|\beta\rangle$ are *not* orthogonal by finding $|\langle\beta|\alpha\rangle|^2$.

   (e) Evaluate $\langle\alpha|\hat{H}|\alpha\rangle$, $\langle\alpha|\hat{H}^2|\alpha\rangle$ and hence ΔH for this state. Show that $\Delta H/\langle H\rangle$ decreases as $|\alpha|$ is increased. [Hint: the calculations should not be long and difficult; if yours are, consider whether you have used the most convenient expression for $\hat{H}$.]

   (f) Evaluate $\langle\alpha|\hat{x}|\alpha\rangle$ and $\langle\alpha|\hat{p}|\alpha\rangle$ and show that $\langle\alpha|\hat{p}|\alpha\rangle = 0$ when α is real while $\langle\alpha|\hat{x}|\alpha\rangle = 0$ when α is imaginary.

   (g) Evaluate Δx and Δp for the state $|\alpha\rangle$ and show that this is a minimum uncertainty state.

   (h) Now, let us consider the time evolution of a coherent state. Suppose that at time $t = 0$, a particle in a harmonic oscillator potential is in the state $|\alpha_0\rangle$. (That is, $\alpha = \alpha_0$ at $t = 0$.) Using the expression for $|\alpha\rangle$ as a superposition of energy eigenstates obtained in part (c), write an expression for the state of the system at all subsequent time $t > 0$. Show that at all subsequent times, the state continues to be of the form (2) for some value of $\alpha \neq \alpha_0$, and therefore continues to be an eigenstate of $\hat{a}$. Determine $\alpha(t)$.

   (i) Using results obtained in various previous parts of this problem, describe how $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{H}\rangle$, Δx and Δp change with time in a coherent state which has $\alpha = \alpha_0$ at $t = 0$, with α_0 real. [Give expressions for all quantities as a function of time. For each quantity which oscillates in time, sketch a graph showing the time dependence; label your sketches sufficiently clearly that the period and amplitude of the oscillation of each quantity are manifest.]

(j) Make a sequence of sketches, showing $|\psi(x)|^2$ vs. x at several times beginning with time $t = 0$, which show how $|\psi(x)|^2$ evolves with time.

2. **Construction of an Orthogonal Coherent State** [15 points]

Consider two coherent states of the harmonic oscillator

$$|\alpha\rangle = N_\alpha \exp(\alpha a^\dagger)|0\rangle\,, \qquad |\beta\rangle = N_\beta \exp(\beta a^\dagger)|0\rangle$$

where α and β are real and N_α and N_β are the normalization constants. Now suppose that β is very close to α, so that $\beta = \alpha + \epsilon$ where $\epsilon \ll 1$. In this problem we will construct a state that is orthogonal to $|\alpha\rangle$ by expanding in ϵ. [For example, expanding N_β gives $N_\beta = N_\alpha(1 - \alpha\,\epsilon + \ldots)$.]

(a) Consider expanding the state $|\beta\rangle$ in ϵ. Write

$$|\beta\rangle = |\alpha\rangle + \epsilon\,|\Psi\rangle + \mathcal{O}(\epsilon^2) \tag{3}$$

and determine the state $|\Psi\rangle$ written as an operator acting on $|\alpha\rangle$.

(b) Using your result in part (a) show that the state $|\Psi\rangle$ is normalized and orthogonal to $|\alpha\rangle$. Recall that $|\alpha\rangle$ satisfies the eigenvalue equation in (1).

(c) Next show that

$$a|\Psi\rangle = \alpha|\Psi\rangle + |\alpha\rangle\,.$$

and use this result to compute $\langle\Psi|\hat{x}|\Psi\rangle$ and $\langle\Psi|\hat{x}^2|\Psi\rangle$. What is the uncertainty in the position, Δx, for the state $|\Psi\rangle$? How do these results compare to those you found for $|\alpha\rangle$ in the previous problem?

(d) Consider the results you derived in part (c) and take $\alpha \gg 1$. Do these properties of $|\Psi\rangle$ behave like you would expect for a coherent state?

(e) Draw a picture of the coordinate space wavefunction $\langle x|\Psi\rangle = \Psi(x)$. Do you recognize this wavefunction? Add $\psi_\alpha(x) = \langle x|\alpha\rangle$ to your picture and explain why the states are orthogonal. [Hint: To draw $\langle x|\Psi\rangle$, consider a $\psi_\alpha(x)$ and $\psi_\beta(x)$ which are close together. Then draw the wavefunction for the difference of the two states. Alternative Hint: Try computing $\langle x|\exp(-ix_0\hat{p}/\hbar)|1\rangle$ and compare the result with $\langle x|\Psi\rangle$. If you use one of the hints show your steps.]

3. **Time Evolution in the Heisenberg Picture** [10 points]

(a) Consider the Hamiltonian $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$ and derive the Heisenberg equations of motion for $\hat{x}_H(t)$ and $\hat{p}_H(t)$. The result you derived in problem set 2, part 6 will be useful. Use your results to obtain Ehrenfest's theorem

$$\frac{d}{dt}\langle\hat{x}\rangle = \frac{\langle\hat{p}\rangle}{m}\,, \qquad \frac{d}{dt}\langle\hat{p}\rangle = -\langle V'(\hat{x})\rangle$$

where here $\langle \hat{x} \rangle = \langle \psi, 0|\hat{x}_H(t)|\psi, 0\rangle = \langle \psi, t|\hat{x}|\psi, t\rangle$ etc.

Note: The equations of Ehrenfest's theorem tell us that the center of a wave packet moves like a classical particle subject to $V(x)$.

(b) Consider the harmonic oscillator and use the Heisenberg equation of motion to evaluate $\frac{d}{dt}\hat{a}_H(t)$ and $\frac{d}{dt}\hat{a}^\dagger_H(t)$. That is, find the time evolution of the Heisenberg picture harmonic oscillator raising and lowering operators. Use your results to obtain $\frac{d}{dt}\hat{x}_H(t)$ and $\frac{d}{dt}\hat{p}_H(t)$ for the harmonic oscillator, and confirm that your results agree with those derived in lecture.

(c) Show that $\hat{a}_H(t) = e^{-i\omega t}\hat{a}$. (Hint: apply the equation you derived for $\hat{a}_H(t)$ to the energy eigenstate $|n\rangle$ and then to an arbitrary state $\sum_n c_n|n\rangle$.)

4. **Properties of the Pauli matrices** [10 points]

The spin operators were defined by $\hat{S}_x = \frac{\hbar}{2}\sigma_1$, $\hat{S}_y = \frac{\hbar}{2}\sigma_2$ and $\hat{S}_z = \frac{\hbar}{2}\sigma_3$. Here the σ_i are the Pauli matrices, and together with the unit matrix we have

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

It is really important to understand the "algebra" of the Pauli matrices, and the following exercises will give you practice using them. The aim of this problem is that you should be as comfortable with these "unit" 2×2 matrices as you are with 1 and i, the basic "units" of the complex numbers. In fact, a number system based on the Pauli matrices is well known among mathematicians as *quaternions*

(a) Show by explicit calculation that

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k, \tag{5}$$

where $i, j = 1, 2, 3$. The summation convention is assumed (so here $\epsilon_{ijk}\sigma_k = \sum_k \epsilon_{ijk}\sigma_k$).

(b) Using the results of part a), show

$$\begin{aligned} [\sigma_i, \sigma_j] &= 2i\varepsilon_{ijk}\sigma_k \\ \{\sigma_i, \sigma_j\} &= 2\delta_{ij} \end{aligned} \tag{6}$$

where $\{A, B\} \equiv AB + BA$ is the *anti-commutator*.

(c) Show

$$\vec{\sigma} \cdot \vec{v}\; \vec{\sigma} \cdot \vec{w} = \vec{v} \cdot \vec{w} + i\vec{\sigma} \cdot (\vec{v} \times \vec{w}), \tag{7}$$

where we use the simplified vector notation $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, *etc.*. Then show

$$\mathrm{Tr}\; \vec{a} \cdot \vec{\sigma}\; \vec{b} \cdot \vec{\sigma}\; \vec{c} \cdot \vec{\sigma} = 2i\, \vec{a} \cdot (\vec{b} \times \vec{c}) \tag{8}$$

Both these results follow directly from part a).

(d) Let $\hat{R}(\vec{a}) = \exp(i\vec{a}\cdot\vec{\sigma})$ and $a \equiv |\vec{a}|$. Using the definition of the exponential function and the properties of the σ-matrices show that

$$\hat{R}(a) = I\cos a + i\vec{\sigma}\cdot\left(\frac{\vec{a}}{a}\right)\,\sin a \tag{9}$$

and verify that $\hat{R}(\vec{a})$ is unitary.

(e) Let

$$\Sigma_k(\lambda) = e^{i\lambda\sigma_3/2}\sigma_k e^{-i\lambda\sigma_3/2}. \tag{10}$$

Show that Σ_1 and Σ_2 can be written

$$\begin{aligned}\Sigma_1(\lambda) &= a_{11}(\lambda)\,\sigma_1 + a_{12}(\lambda)\,\sigma_2 \\ \Sigma_2(\lambda) &= a_{22}(\lambda)\,\sigma_2 + a_{21}(\lambda)\,\sigma_1\end{aligned} \tag{11}$$

and find the coefficients, a_{ij}, which should be simple functions of λ.