Quantum Physics II (8.05) Fall 2004
Assignment 10

Massachusetts Institute of Technology
Physics Department
November 11, 2004

Due November 18, 2004
Thur., 7:00pm

This week we continue our study of three dimensional quantum mechanics problems. In particular we will use an operator method, known as the "factorization method" or "supersymmetric method" to solve the radial equation. The reading on this method is given below, however problems on this material will only be on next week's set (#11). This week's problem set covers bound and unbound solutions to the radial equation.

Reading Assignment for week 10 of the course

- (continued from last week) Griffiths §4.1, §4.2, §4.3.
- A nice summary of the material on the qualitative behavior of the solutions to the radial equation can be found in Shankar (Second Edition) pp 339-345.
- The "factorization method" is discussed in Ch.6 and Ch.8 of Ohanian.
  Ohanian sets up his notation and proves a general theorem in §6.2.
  (We will *not* cover the details of this particular theorem in lecture.)
  Ch.8 contains three important examples of the factorization method, i) the isotropic harmonic oscillator, ii) hydrogen, and iii) the free particle in spherical coordinates.

Problem Set 10

1. **Half Integer Spherical Harmonics Are Not Allowed** [10 points]

   We have seen that the commutation relations for angular momentum operators allow $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ However, closer inspection shows that it is not possible to formulate the quantum mechanics of orbital angular momentum consistently in the Hilbert spaces with half-integer orbital angular momentum.

   (a) Assume that $\ell = 1/2$ is allowed. Then spherical harmonics for $\ell = 1/2$ should exist. Find the spherical harmonic for $\ell = m = 1/2$ (as we found the $Y_{\ell m}$ for integer ℓ) by using the equations:

$$\begin{aligned} L_+\Big|\ell = \frac{1}{2}, m = \frac{1}{2}\Big\rangle &= 0 \\ L_3\Big|\ell = \frac{1}{2}, m = \frac{1}{2}\Big\rangle &= \frac{\hbar}{2}\Big|\ell = \frac{1}{2}, m = \frac{1}{2}\Big\rangle \end{aligned}$$

Your answer should be of the form:

$$Y_{\frac{1}{2},\frac{1}{2}} = N(\sin\theta)^a e^{ib\phi}$$

Find a, b, and show that $Y_{\frac{1}{2},\frac{1}{2}}$ is normalizable.

(b) Repeat part a) for $\ell = \frac{1}{2}$, $m = -\frac{1}{2}$.

(c) Now show that the operator L_- applied to $Y_{\frac{1}{2},\frac{1}{2}}$ *does not* give $Y_{\frac{1}{2},-\frac{1}{2}}$.
[Since the only ingredients that went into the derivation of the relation $L_-|\ell, m\rangle \propto |\ell, m-1\rangle$ were the commutation relations of the L_k and the fact that they are Hermitian, we are forced to conclude that we cannot implement orbital angular momentum in terms of Hermitian operators on the Hilbert (sub)space spanned by $\ell = \frac{1}{2}, m = \pm\frac{1}{2}$.]

(d) Suppose you were to construct the spherical harmonic for $\ell = 3/2$, $m = 3/2$ in the manner of part a). You would be led to the result,

$$Y_{3/2,3/2} = c\,(\sin\theta)^{3/2} e^{3i\phi/2} \ ,$$

which satisfies $L_+ Y_{3/2,3/2} = 0$. Generate $Y_{3/2,-3/2}$ by repeated use of L_-. Show that the state you obtain (i) does not satisfy $L_- Y_{3/2,-3/2} = 0$ and (ii) is not even normalizable.

Comments: Although half-integer *orbital* angular momentum is not possible, particles with half-integer angular momentum do really exist. Their intrinsic angular momentum, or "spin" is not a property of their motion in space. There is no representation of the spin angular momentum operators in terms of θ and ϕ, there are no "wavefunctions" in θ and ϕ for spin, and so no problem! One usually reserves the notation L^2, L_z and ℓ only for orbital angular momentum, and thus for integer-valued ℓ, and will adopt this notaion. When discussing angular momentum in general, including the possibility of orbital angular momentum and/or spin, we will call the operators J^2 and J_z, and denotes the quantum numbers by j and m, so $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ where j can be integer valued or half-integer valued.

2. **Qualitative behavior of the radial wavefunction** [15 POINTS]

Consider a particle of mass m moving under the influence of an attractive "Yukawa" potential, $V(r) = -ge^{-\alpha r}/r$.

(a) What is the "effective" potential for the radial Schödinger equation? Write the Schrödinger equation for the radial wavefunction, $u(r)$.

(b) Define a dimensionless radial variable $x \equiv \alpha r$ and rewrite the radial equation so it has the form,

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} - g'\frac{e^{-x}}{x}\right] u(x) = \lambda u(x)$$

Relate λ and g' to the original parameters of the problem.

(c) Show graphically that the parameter g' can be chosen so this potential has a) no bound states, or b) many bound states (for any ℓ). Note that it is the interplay between the interaction $V(r)$ and the angular momentum barrier, $\hbar^2\ell(\ell+1)/2mr^2$ that determines whether and how many bound states occur.

(d) Suppose the parameters are such that there are four distinct, bound energy levels with $\ell = 2$ for this problem. Sketch **as accurately as you can** the wavefunction of the **second most tightly bound energy level with** $\ell = 2$. **Do not solve the Schrödinger equation numerically. Instead use your physical understanding of the solutions to the equation.** You should include: the behavior as $x \to 0$, the behavior as $x \to \infty$, the correct number of nodes, the relative magnitude of the wavefunction at large, small, and intermediate x.

3. **Solving the Hydrogen-like Atom** [19 POINTS]

The problem of an electron bound to a nucleus of charge Ze can be rewritten in a simple form by choosing "natural" units of length. The aim of this problem is to go through the solution of the Schrödinger equation in (a), (b), (c), (d), essentially in the same way that I did in lecture, and then use the solution to answer the problems in parts (e), (f), (g), and (h).

Consider an electron bound to a nucleus of charge Ze by the Coulomb potential, $V(r) = -\frac{Ze^2}{r}$. μ is the reduced mass: $1/\mu = 1/m_e + 1/M$. We are using Gaussian units.

The Schrödinger equation for the radial wavefunction, $u(r)$, reads

$$\left\{-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2} - \frac{Ze^2}{r}\right\} u(r) = -Bu(r). \qquad (1)$$

(a) Define a scaled length variable, $r \equiv bx$. Show that you can choose b so that the Schrödinger equation can be written

$$\left\{-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} - \frac{1}{x}\right\} u(x) = -\kappa^2 u(x). \qquad (2)$$

What are b and κ in terms of Z, e, μ, $\hbar$ and B?

(b) Show that at small x, $u(x)$ behaves approximately like $x^{\ell+1}$ and that at large x $u(x)$ behaves like $\exp(-\kappa x)$.

(c) Based on the result of the part (b), the simplest guess one could make for $u(x)$ would be $u_\ell(x) = A_\ell x^{\ell+1}\exp(-\kappa x)$. Show that this in fact exactly solves (2), and find the eigenvalue κ. Inverting all the definitions of variables find an expression for the binding energy of this state.

(d) Explain (words or pictures only, please) why the state you found in part (c) must be the lowest energy (most tightly bound) state with angular momentum ℓ.

(e) Find $\langle r\rangle$, $\langle r^2\rangle$ and $\langle 1/r\rangle$ for the ground state of the hydrogen-like atom. Express your answers in terms of the Bohr radius $a_0 = \hbar^2/(me^2)$.

(f) For an energy eigenstate of hydrogen, $E_{n,\ell} = \langle H\rangle = \langle \frac{p^2}{2m} - \frac{Ze^2}{r}\rangle$. The virial theorem says $\langle \frac{p^2}{2m}\rangle = \frac{1}{2}\langle \frac{Ze^2}{r}\rangle$ for a Coulomb potential. To check the virial theorem use it together with your result of part (e) to calculate $E_{n,\ell}$ for the ground state.

(g) Calculate the expectation value of $p^2/(m^2c^2)$ for the hydrogen ($Z = 1$) ground state. Call your answer α^2. Evaluate α numerically and show that it is dimensionless and equals approximately 1/137.

[Comments: This result illustrates that the motion of the electron in hydrogen is not relativistic. Here α is a dimensionless parameter measuring the strength of electromagnetic forces, since it involves only e and the fundamental constants $\hbar$ and c. Note that a "hydrogen like atom" does not become more or less relativistic as the mass of the "electron" gets smaller or larger. For example the root mean square velocity of the charged particles in positronium (an atom made of a positron (e^+) and an electron) or in muonium (an atom made of a proton and a muon with $m_\mu = 207\, m_e$), is still α.]

(h) **Beta Decay of Tritium**

An electron is in the ground state of tritium. Tritium is just like hydrogen, except that its nucleus consists of one proton and two neutrons. The tritium nucleus is unstable. After the decay (which can be thought of as instantaneous for our purposes) the nucleus is a ^{3}He nucleus made of two protons and one neutron. Thus, Z has changed from 1 to 2. [The reduced mass μ has changed by a tiny amount, but you can ignore this.] The statement that the decay is "instantaneous" means that immediately after the decay, the wavefunction of the electron in the atom is unchanged from what it was immediately before the decay. The electron is, however, no longer in an energy eigenstate. Calculate the probability that the electron is in the ground state of ^{3}He. What is the probability that it is in some state with $\ell = 0$? [No calculation should be required to answer the latter question.]

[Note that the nuclear decay produces an electron and an antineutrino. These exit the system quickly enough that we can neglect them. This is (part of) what I mean when I say that the decay is instantaneous.]

4. **Phase Shifts** [16 points]

States with positive energy are not bound, and their energies are not quantized. They are important as solutions to the scattering problem. When we scatter a particle of momentum k from a central potential, we can observe the phase shifts $\delta_\ell(k)$ for each ℓ. (We will see how in 8.06.) This problem reviews the definition of the phase shift and then asks you to derive it in a couple simple cases.

The radial equation for a solution with energy E and angular momentum ℓ is

$$-\frac{d^2}{dz^2}u_\ell(z) + \frac{\ell(\ell+1)}{z^2}u_\ell(z) + \frac{2m}{\hbar^2 k^2}V(z)u_\ell(z) = u_\ell(z) \tag{3}$$

where $k^2 = 2mE/\hbar^2$, $z = kr$ and $u_\ell(0) = 0$.

(a) First consider the case $V = 0$. Call these solutions to the free Schrödinger equation $u^0_\ell(z)$ where $z = kr$. These functions are the "Spherical Bessel Functions" that were derived in lecture using the supersymmetric method. Here show that $u^0_0(z) \propto \sin z$, and the recursion relation $u^0_{\ell+1} = (\frac{\ell+1}{z} - \frac{d}{dz})u^0_\ell$, to show

$$\lim_{z\to\infty} u^0_\ell(z) \sim \sin\left(z - \frac{\ell\pi}{2}\right) \tag{4}$$

This explains why the *phase shift* for an arbitrary potential, V is defined by

$$\lim_{z\to\infty} u_\ell(z) \sim \sin\left(z - \frac{\ell\pi}{2} + \delta_\ell(k)\right) \tag{5}$$

(b) Consider the potential $V(r) = -V_0$ for $r < b$ and $V = 0$ for $r > b$ (with $V_0 > 0$). Solve the Schrödinger equation for $\ell = 0$ and $k^2 > 0$ for this potential. Remember that both u and u' must be continuous at any discontinuity of V.

Obtain an equation for the phase shift $\delta_0(k)$ and plot your result for V_0 small enough that there is no bound state.

Discuss the behavior of the phase shift in the limit of high energies ($k^2 \gg mV_0/\hbar^2$) and low energies ($k^2 \ll mV_0/\hbar^2$).

[Answer: the equation obeyed by the phase shift is $k\cot(kb+\delta_0(k)) = q\cot(qb)$, where $q^2 = 2m(E+V_0)/\hbar^2$. $\delta_0(k)$ vanishes at $k=0$, rises linearly (at first) with k, reaches a maximum, and then falls back to zero at large k. I leave to you deriving the answer, plotting it, and discussing its behavior.]

(c) Consider the potential $V(r) = \infty$ for $r < b$ and $V = 0$ for $r > b$. Again, calculate the $\ell = 0$ phase shift. [This potential corresponds to an impenetrable sphere.]