

Quantum Physics II (8.05) Fall 2004

Assignment 7

Massachusetts Institute of Technology
Physics Department
October 19, 2004

Due **FRIDAY** *October 29, 2004*
7:00pm

This week we will finish the discussion of neutrino oscillations and then take up kaons.

Reading Assignment for week 7 of the course

- (continued from last week) Recommended: Feynman, Vol. III, §8.6 The Ammonia Molecule. (Also §9 The Ammonia Maser, although we will not cover this material in lecture.)

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Problem Set 7

1. The Ammonia Molecule (20 points)

Consider an ammonia molecule placed in an electric field ε . The Hamiltonian is given by

$$\hat{H} = \begin{pmatrix} E_0 + \mu\varepsilon & -\Delta \\ -\Delta & E_0 - \mu\varepsilon \end{pmatrix}$$

in the $\{|+\rangle, |-\rangle\}$ basis, where μ is the electric dipole moment. [Note: Here the states $|+\rangle$ and $|-\rangle$ are those in which the nitrogen atom is either above or below the plane of the hydrogens. These states are evidently not energy eigenstates.]

In this problem, you will find and study the energy eigenstates and energy eigenvalues of the Hamiltonian \hat{H} .

- (a) Show that the Hamiltonian may be written as

$$\hat{H} = E_0 \hat{1} + a (-\sin(2\theta)\sigma_1 + \cos(2\theta)\sigma_3)$$

for some a and θ . Specify a and $\tan(2\theta)$ in terms of variables in the original Hamiltonian.

- (b) Use an identity derived on a past problem set to show that the Hamiltonian can be rewritten as

$$\hat{H} = E_0 \hat{1} + a (\exp(i\theta\sigma_2)\sigma_3 \exp(-i\theta\sigma_2)) .$$

- (c) Define $\hat{U} = \exp(i\theta\sigma_2)$. Show that $|I\rangle = \hat{U}|+\rangle$ and $|II\rangle = \hat{U}|-\rangle$ are the eigenstates of the Hamiltonian. What are the corresponding energy eigenvalues? Evaluate $\hat{U}^\dagger \hat{H} \hat{U}$.

[Note: \hat{U} is unitary — that's why I called it \hat{U} — but it is *not* the time evolution operator for a system with Hamiltonian \hat{H} , ie. $\hat{U} \neq \exp(-i\hat{H}t/\hbar)$.]

- (d) Show that the energy eigenstates may be written as $|I\rangle = \cos\theta|+\rangle - \sin\theta|-\rangle$ and $|II\rangle = \sin\theta|+\rangle + \cos\theta|-\rangle$.
- (e) Consider the limit in which $\mu\varepsilon = 0$. To what θ does this correspond? What are the eigenstates and eigenvalues of the Hamiltonian in this limit?
- (f) In the opposite limit, in which $\mu\varepsilon$ is much larger than Δ , what happens to θ ? What are the energy eigenstates and eigenvalues in this limit? Interpret these results.
- (g) Take $E_0 = 0$ and plot the two energy eigenvalues versus ε . Include on your plot the limits you considered in (e) and (f) by labelling their energy eigenvalues and also indicating the corresponding eigenstates.

2. Practical Neutrino Oscillations [25 points]

- (a) Take the neutrino wave function from eq. (4) of the “supplementary notes on neutrino oscillations and kaon physics” and show that the probability to observe an electron neutrino in an initially pure ν_μ beam is given by eq. (5), that is

$$P_{\nu_e}(L) = \sin^2 2\theta \sin^2 \frac{c^3 \Delta m^2 L}{4\hbar E} .$$

- (b) Substitute fundamental constants and show that this reduces to eq. (6) of the notes, namely

$$P_{\nu_e}(L) = \sin^2 2\theta \sin^2 \frac{1.27 \Delta m^2 c^4 L}{E} ,$$

where $\Delta m^2 c^4$ is the difference between the squares of the mass eigenvalues, measured in eV^2 ; the neutrino energy E is measured in MeV and L is measured in m .

- (c) Suppose that we caricature the Los Alamos experiment, and pretend that all the neutrinos have $E = 30 MeV$, all travel $L = 30 m$, and suppose that the result of the experiment is $P_{\nu_e} = 0.003$, completely ignoring the experimental uncertainties. This caricatured result could be explained by many different choices for $(\Delta m^2 c^4)$ and $\sin^2 2\theta$, which form a curve in the $(\Delta m^2 c^4) - \sin^2 2\theta$ plane. Sketch this curve. (You should consider using a log scale for $\Delta m^2 c^4$, ranging from $0.01 eV^2$ to $20 eV^2$, and a log scale for $\sin^2 2\theta$ ranging from 0.001 to 1 . You may either use some computer package to make a graph, or you may just find sufficiently many points on the curve “by hand” that you can make a sketch.)
- (d) Suppose that $\sin^2 2\theta = 1$. What is the smallest value of $\Delta m^2 c^4$ which explains the Los Alamos data? (There is no evidence at all favoring this point on the curve in part (c) over any other; I am just simplifying the following problem for you.) Let us now design an experiment to confirm or refute the hypothesis that $\sin^2 2\theta = 1$ and $\Delta m^2 c^4$ is given by the value which you have just found. Fermilab will be happy to supply you with a beam of muon neutrinos with an energy E of about $500 MeV$. What is the shortest baseline, L , which you could use so that after travelling this distance, the neutrinos are *all* electron neutrinos? Now, however, you figure out that if you put your detector farther than $1 km$ away from the neutrino source, the beam which Fermilab makes will be too spread out for you to see enough neutrinos to do a good experiment. So, you decide to put your detector at $L = 1 km$. How small a P_{ν_e} will you be looking for? [These numbers are at least approximately correct for an experiment called MiniBooNE being planned, using the Fermilab “booster” as the source of the protons which make the pions which make the neutrinos.]
- (e) Super-Kamiokande observes atmospheric neutrinos with a range of energies, but for this problem let us pretend that $E = 3.5 GeV$. Take as a given that

their analysis of their data is correct and assume that $\Delta m^2 c^4 = 2.4 \times 10^{-3} \text{ eV}^2$ and $\sin^2 2\theta = 1$. How many times does a ν_μ produced on the far side of the planet oscillate from ν_μ to ν_τ and back to ν_μ as it travels through the center of the earth from where it was produced to Japan? Answer the same question, but this time for a neutrino with $E = 6 \text{ GeV}$. [Conclude that since Super-Kamiokande does not distinguish between $E = 3.5 \text{ GeV}$ and $E = 6 \text{ GeV}$, the results for these energies and all in between should be averaged, leading to the prediction that the observed $|\nu_\mu\rangle$ flux should be approximately 50% that predicted which would be observed if there were no oscillations.] Now, suppose that you wish to design an experiment to confirm that $\Delta m^2 c^4 = 2.4 \times 10^{-3} \text{ eV}^2$ and $\sin^2 2\theta = 1$. Fermilab is willing to supply you with a beam from its “main injector” which is much more energetic and better collimated than the beam from its “booster”. Suppose that these muon neutrinos have $E = 4 \text{ GeV}$. How far away should your detector be such that there are no muon neutrinos in the beam when the beam reaches your detector? Again, this is too far away to be practical. The Soudan mine is 730 km away, which will have to do. By what fraction will the muon neutrino flux be reduced at the detector? (This experiment is called MINOS.)

3. Pollution from ν_e 's in a ν_μ -disappearance experiment [15 points]

Suppose the electron and muon neutrinos are linear superpositions of mass eigenstates, ν_1 and ν_2 , as given in eq. (1) of the supplementary neutrino oscillation notes. When the positively charged pion (mass $140 \text{ MeV}/c^2$) decays, it decays most of the time into a μ^+ and a muon neutrino, but a fraction $p = 1.2 \times 10^{-2}\%$ of the time it decays into a positron (e^+) and an electron neutrino.

- (a) Suppose that at time $t = 0$ you have energetic neutrinos (say energies $\sim 10 \text{ GeV}$) formed by π^+ decay. Each pion decay produces either a muon neutrino or an electron neutrino. (You can tell which by observing whether a muon or an electron is produced in the decay.) Assume that the neutrino beam is “dilute”, in the sense that each individual neutrino is described by a wave packet that is well separated in space from the wave packet for the neutrino ahead and behind. This means that the evolution of each neutrino subsequent to its production is independent of that of the other neutrinos. What is the probability of observing a muon neutrino as a function of the distance from the source L ? (Your answer will depend on p , the mixing angle θ , the mass difference $m_1^2 - m_2^2$, and the neutrino energies E_{ν_e} and E_{ν_μ} .)
- (b) Now, you are at a distance L observing neutrinos of a fixed energy E . Consider your result from part (a). For what values of θ and $m_1^2 - m_2^2$ can the electron neutrino “pollution” parameterized by the small probability p be neglected? For what values of θ and $m_1^2 - m_2^2$ does the fact that $p \neq 0$ modify the outcome even though p is small?