

Quantum Physics II (8.05) Fall 2004

Assignment 12

Massachusetts Institute of Technology
Physics Department
November 23, 2004

*Due FRIDAY December 3, 2004
7pm*

This week we tackle the problem of combining the angular momentum of different parts of a system (eg. spin and orbital angular momentum) into the total angular momentum. Two different complete sets of commuting operators turn out to be useful, and one can transform between the two eigenbases with a unitary transformation. The elements in the matrix for the unitary transformation are called Clebsch-Gordan coefficients.

Reading Assignment for week 12 of the course

- (continued from last week) Griffiths §4.4.3, Addition of Angular Momenta
- Griffiths' treatment of this subject is brief. For further reading on the Addition of Angular Momenta see: Ohanian, §9.3, Shankar §15.1 and 15.2, and Cohen-Tannoudji, Vol.II, §X .
- In the last week of the course we will discuss Identical Particles. If you would like to get ahead on your reading, then read: Griffiths §5.1 and for additional reading see also Ohanian §9.4, Cohen-Tannoudji, §XIV, and Shankar §10.

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Problem Set 12

1. CSCOs for Spin-Orbit Coupling [12 points]

Including the spin-orbit coupling, the electronic Hamiltonian for hydrogen with no external fields is

$$\mathcal{H} = \mathcal{H}_0 + \frac{2\mu_B^2}{r^3} \frac{\vec{L} \cdot \vec{S}}{\hbar^2},$$

where $\mathcal{H}_0 = \vec{p}^2/2m - e^2/r$ as usual. \vec{L} and \vec{S} are the electron orbital and spin angular momentum (operators). μ_B is the Bohr magneton. For this problem, simply take this as a definition of \mathcal{H} .

(a) Evaluate the commutators

$$[\mathcal{H}, L^2], \quad [\mathcal{H}, L_z], \quad [\mathcal{H}, S^2], \quad [\mathcal{H}, S_z], \quad [\mathcal{H}, J^2], \quad [\mathcal{H}, J_z]$$

where $\vec{J} = \vec{L} + \vec{S}$ is the total angular momentum. What is the largest set of these operators including \mathcal{H} that is mutually commuting?

(b) Consider states whose radial wave functions are eigenstates of the radial part of \mathcal{H}_0 with principal quantum number $n = 1$ or $n = 2$. What values of ℓ , s , j and m_j are possible? If we include the effects of the spin-orbit coupling, these states are not all degenerate. How many distinct energy levels are there? What is the degeneracy of each? Draw a diagram showing the levels in order of energy. [i.e. determine which levels have higher energy and which have lower energy, but do not calculate the magnitude of the energy differences between levels.]

(c) Now turn on an external magnetic field $\vec{B} = B\vec{e}_z$, so that the term

$$\mathcal{H}_B = \frac{\mu_B}{\hbar} \vec{B} \cdot (\vec{L} + g\vec{S})$$

is added to the Hamiltonian, where $g \simeq 2$. Repeat part (a).

2. Total Spin States [15 points]

(a) Suppose you are given two (non-identical) particles, each with angular momentum 1, so $j_1 = j_2 = 1$. How many distinct angular momentum eigenstates does this system possess? What are the possible values of the total angular momentum quantum number (J) of this system? [In this problem we use a capital J to denote the total angular momentum, to avoid confusion with the various small j 's.]

(b) Repeat (a) for two particles, each with angular momentum j , so $j_1 = j_2 = j$.

(c) Suppose you are given *three* (non-identical) particles, each with angular momentum $j = 1$. How many distinct angular momentum eigenstates does this system possess? What are the possible values of the total angular momentum (J) of this system?

(d) Repeat (c) but with $j = 3/2$.

[Hint: A general way of solving such problems involves counting and analysing the possible m -values. Here's how it works for three particles with $j = 1/2$. In a basis where j_{1z} , j_{2z} , and j_{3z} are diagonal with eigenvalues m_1 , m_2 , and m_3 (which can be either $\frac{1}{2}$ or $-\frac{1}{2}$), states can be labelled by the eigenvalues of m_1 , m_2 , and m_3 . We can make a table of all the states: Notice that $M = m_1 + m_2 + m_3$ is the eigenvalue

State	m_1	m_2	m_3	$M = m_1 + m_2 + m_3$
1	1/2	1/2	1/2	3/2
2	1/2	1/2	-1/2	1/2
3	1/2	-1/2	1/2	1/2
4	-1/2	1/2	1/2	1/2
5	1/2	-1/2	-1/2	-1/2
6	-1/2	1/2	-1/2	-1/2
7	-1/2	-1/2	1/2	-1/2
8	-1/2	-1/2	-1/2	-3/2

Table 1: States of three spin 1/2 particles.

of $J_z = j_{1z} + j_{2z} + j_{3z}$. Each state of definite M must be part of a multiplet of definite total angular momentum J (with M values ranging from $-J$ to J). Looking at the list of M values we see that the M values corresponding to $J = 3/2$, $J = 1/2$ and $J = 1/2$ are represented. So we conclude that the total angular momentum states of three spin-1/2 particles are $J = 3/2$ and two copies of $J = 1/2$. Another way to do this problem is to use the composition law for two angular momenta *twice*. Schematically,

$$1/2 \times 1/2 \times 1/2 = [0 + 1] \times 1/2 = [0 \times 1/2] + [1 \times 1/2] = 1/2 + 1/2 + 3/2 \quad (1)$$

End of hint.]

3. **Based on Gasiorowicz §15, problem 4** [5 points]

A particle of spin-1 moves in a central potential of the form

$$V(r) = V_1(r) + \frac{\vec{S} \cdot \vec{L}}{\hbar^2} V_2(r) + \frac{(\vec{S} \cdot \vec{L})^2}{\hbar^4} V_3(r) \quad (2)$$

What are the values of $V(r)$ in the states with $j = \ell + 1$, $j = \ell$, and $j = \ell - 1$?

[Hint: Use $\vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L} \cdot \vec{S}$.]

4. **Clebsch-Gordan Coefficients for $\ell = 1$ and $s = 1/2$.** [20 points]

Consider a spin $1/2$ particle in a state with orbital angular momentum $\ell = 1$. The aim of this problem is to calculate the Clebsch-Gordan coefficients that allow us to construct states of definite total angular momentum from simultaneous eigenstates of spin and orbital angular momentum. Label the eigenstates in the “uncoupled basis” (ie eigenstates of L^2 , L_z , S^2 and S_z) by $|\ell m_\ell s m_s\rangle$. Label the states in the “coupled basis” (ie eigenstates of J^2 and J_z) by $|j, m\rangle$.

- Find the state with maximum j and m ($= j_{max}$) in terms of the $|\ell m_\ell s m_s\rangle$ states.
- Use $J_- = L_- + S_-$ to generate all the $|j_{max}, m\rangle$ states.
- Use orthonormality to find the state $|j_{max} - 1, j_{max} - 1\rangle$.
- Use J_- to generate all the states $|j_{max} - 1, m\rangle$.
- Repeat steps (c) and (d) for smaller j 's as many times as necessary.
- What is the expectation value of L_z in the state with $j = 1/2$, $m = 1/2$? What is the expectation value of S_z in this state?
- Suppose that this particle moves in an external magnetic field in the z -direction, $\vec{B} = B\vec{e}_z$. Assume the particle is an electron, and take $g = 2$. The Hamiltonian describing the interaction of the electron with the field is

$$H_B = \frac{\mu_B}{\hbar} \vec{B} \cdot (\vec{L} + 2\vec{S}) .$$

What is $\langle H_B \rangle$ in each of the eigenstates $|j, m\rangle$?

- For the eigenstate $j = 1/2$, $m = 1/2$, what are the possible values of the magnetic energy and what are their probabilities?

[For those of you who want more practice, work out the Clebsch-Gordan coefficients for the addition of orbital angular momentum ℓ and spin 1. I decided not to put it on the problem set, as it involves a fair amount of algebra. It is, however, very good practice.]

5. **Practice with the Table of Clebsch-Gordan Coefficients** [8 points]

This problem is from Griffiths (4.36), and you should use his Table 4.8 on page 188.

- A particle of spin-1 and a particle of spin-2 are at rest in a configuration such that the total spin is 3, and its z component is \hbar . If you measured the z component of the angular momentum of the spin-2 particle, what values might you get, and what is the probability of each one?
- An electron with spin down is in the state $\psi_{n\ell m} = \psi_{510}$ of the hydrogen atom. If you could measure the total angular momentum squared of the electron alone (*not* including the proton spin), what values might you get, and what is the probability of each?