

Quantum Physics II (8.05) Fall 2004

Assignment 9

Massachusetts Institute of Technology
Physics Department
November 4, 2004

*Due FRIDAY November 12, 2004
7:00pm*

Quantum mechanics in three dimensions. For central potentials $\{H, \vec{L}^2, L_z\}$ are a CSCO, and so the problem reduces to i) a kinematic “angular” part for finding eigenstates of $\{\vec{L}^2, L_z\}$, and ii) a “radial” dynamic part for finding energy eigenstates of H . Only ii) depends on the potential. This week’s problems are on the separation of variables and the operator structure of angular momentum. Next week we’ll move on to the physics of the radial equation.

Reading Assignment for week 9 of the course

- Griffiths §4.1, §4.2, §4.3.
- For further reading on angular momentum (some with more focus on operators than Griffiths), consult one or more of the following:

Ohanian Ch.7.

Shankar (Second Edition) §12.5. If you want to read more about the importance of angular momentum in the theory of rotations you can read the material in Shankar surrounding §12.5.

Cohen-Tannoudji, Ch. VI pp 643-676.

Gasiorowicz (Third Edition) §7.

Problem Set 9

1. Commutators for a Central Potential [8 points]

Recall that the orbital angular momentum operators are $L_j = \varepsilon_{jkm} x_k p_m$. In three dimensions the Hamiltonian for a particle in a central potential $V(r)$ is

$$H = \frac{\vec{p}^2}{2m} + V(r).$$

In lecture we showed that this Hamiltonian can also be written as

$$H = \frac{1}{2m} \left[\frac{1}{r^2} (\vec{r} \cdot \vec{p})^2 - \frac{i\hbar}{r^2} (\vec{r} \cdot \vec{p}) \right] + \frac{1}{2mr^2} \vec{L}^2 + V(r).$$

In this problem you will show that $[L_j, H] = 0$ using this second form of H .

- First show $[L_j, \vec{r} \cdot \vec{p}] = 0$.
- Now let $f(r)$ be any function of the radius operator in three dimensions ($r = |\vec{r}|$). Show $[L_j, f(r)] = 0$. [Hint: First show $[L_j, r] = 0$, then generalize.]

On your last problem set you showed that $[L_j, \vec{L}^2] = 0$. Together with the results in (a) and (b) this implies the desired result, $[L_j, H] = 0$.

2. Parity in 3-dimensions [12 points]

On problem set 3 you encountered parity in one dimension and you may want to review the solution to that problem before doing this one. In this problem you will extend the analysis to three dimensions and work out how parity acts on the angular momentum eigenstate $|\ell, m\rangle$. The parity operator Π is defined by

$$\Pi|\vec{r}\rangle = |-\vec{r}\rangle. \quad (1)$$

and is hermitian and unitary, $\Pi^\dagger = \Pi = \Pi^{-1}$, so $\Pi^2 = I$.

- Using results you know about the one dimensional $|x\rangle$ and $|p\rangle$, determine results for $\langle \vec{r} | \vec{p} \rangle$ and $\langle \vec{p} | \vec{r} \rangle$. Derive a result for Π acting on $|\vec{p}\rangle$ that is analogous to (1).
- An anticommutator is defined by $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. Show that $\{\Pi, r^i\} = 0$ and $\{\Pi, p^i\} = 0$ where r^i and p^i are the coordinate and momenta operators. Next show that

$$[\Pi, L^i] = 0 \quad (2)$$

where L^i are the angular momentum operators.

- Now that we know (2), recall that the states $|\ell, m\rangle$ were complete non-degenerate eigenkets for \vec{L}^2, L_z . Explain why this implies that

$$\Pi|\ell, m\rangle = \lambda_{\ell, m}|\ell, m\rangle \quad (3)$$

for some real eigenvalues $\lambda_{\ell, m}$. Show that $(\lambda_{\ell, m})^2 = 1$.

- (d) Equation (2) implies that $[\Pi, (L_{\pm})^k] = 0$ for integers $k > 0$ where L_{\pm} are the raising/lowering operators defined in lecture. Use this result to prove that $\lambda_{\ell, m}$ is independent of m , so that you can write

$$\Pi|\ell, m\rangle = \lambda_{\ell}|\ell, m\rangle.$$

Hint: To start yourself off, consider what $[\Pi, L_+] = 0$ implies when it acts on $|\ell, 0\rangle$.

- (e) What is $\langle \vec{r} | \Pi = \langle -\vec{r} |$ in spherical polar coordinates? In lecture we showed that

$$\langle \theta, \phi | \ell, \ell \rangle = Y_{\ell, \ell}(\theta, \phi) = c_{\ell} (\sin \theta e^{i\phi})^{\ell}$$

with normalization constants c_{ℓ} . Use this wavefunction to derive that $\lambda_{\ell} = (-1)^{\ell}$. You're done! The final result is

$$\Pi|\ell, m\rangle = (-1)^{\ell}|\ell, m\rangle.$$

3. Measurement of Angular Momentum [15 points]

A quantum particle is known to have total angular momentum one, *i.e.* $\ell = 1$.

- (a) Use the eigenstates of L_3 (denote them as $|\ell, m\rangle = |1, m\rangle$) as a basis and find the *matrix* representation of the three operators, L_1 , L_2 and L_3 in this three dimensional subspace.
[Hint: You know the action of the operators L_3 , and L_{\pm} on the states $|1, m\rangle$, and you know the relation between L_{\pm} and L_1 and L_2 .]
- (b) Verify that the matrices you found in part a) satisfy the commutation relation $[L_1, L_2] = i\hbar L_3$.
- (c) Find expressions for the eigenstates $|1, m_1 = +1\rangle$, $|1, m_1 = 0\rangle$, and $|1, m_1 = -1\rangle$ as superpositions of eigenstates of L_3 . For example, write

$$|1, m_1 = +1\rangle = \sum_{m_3} C(m_3) |1, m_3\rangle \tag{4}$$

and determine $C(m_3)$. [Hint: You have got a matrix representation for L_1 . What do the eigenvectors of that matrix have to do with this problem?]

- (d) If a particle is in the state $|1, m_1 = 1\rangle$ and a measurement is made of the L_3 component of its angular momentum, what are the probabilities of the outcomes $m_3 = +1, 0, -1$?
- (e) A particle is in the state $|1, m_1 = 1\rangle$. The L_3 component of its angular momentum is measured and the result $m_3 = +1$ is obtained. Immediately afterwards, the L_1 component of angular momentum is measured. Explain what results are obtained with what probability. Explain why $m_1 = +1$ is not obtained with probability unity.

4. **Particle on a Sphere** [10 points]

Consider a spin-less particle with mass M confined to the surface of a sphere with radius $r = 1$. The particle can move freely on the sphere, but cannot deviate from $r = 1$.

- (a) Write the Hamiltonian, $H = L^2/(2Mr^2) = L^2/(2M)$, in the position representation in spherical coordinates. Calculate the energy eigenvalues and determine the energy eigenstates. Draw the energy level diagram, and specify the degeneracy of each level. Does the Hamiltonian constitute a complete set of commuting observables (CSCO) by itself? If the answer is no, give an example of a CSCO.
- (b) Suppose that the position representation of the state $|\psi\rangle$ of the particle is

$$\langle \vec{r} | \psi \rangle = \psi(\vec{r}) = \frac{1}{\sqrt{8\pi}} + \sqrt{\frac{15}{32\pi}} \cos 2\phi \sin^2 \theta . \quad (5)$$

Is this state properly normalized? What is the probability of finding the particle at some position with θ between $\pi/4$ and $\pi/2$?

- (c) Suppose the particle is in the state (5). Suppose that you measure the energy, the square of the total angular momentum, and the z -component of the angular momentum. What are the possible results? What is the probability for obtaining each result?

5. **“Cartesian Harmonics”** [10 points]

The spherical harmonics with a given ℓ can be written as linear combinations of terms of the form $x^a y^b z^c / r^\ell$ with $a + b + c = \ell$. This gives some insight into the fact that the $Y_{\ell m}$ with $\ell = 0, 1, 2, \dots$ and $m = -\ell, -\ell + 1, \dots, \ell - 1, \ell$ form a complete set for expanding functions of angles.

- (a) Express the spherical harmonics for $\ell = 0, 1, 2$ in terms of $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$.
- (b) A particle in a spherically symmetric potential is described by the wave function

$$\psi(x, y, z) = C(x^2 - y^2 - z^2)e^{-\alpha r^2} .$$

You do not need to know C or α to answer the following:

- i. What is the probability that a measurement of the square of the angular momentum yields 0?
- ii. What is the probability that it yields $6\hbar^2$?
- iii. If ℓ is found to be 2, what are the relative probabilities for a measurement of L_z to yield $m = 2, 1, 0, -1, -2$?

6. Rotation Operator [5 points]

Show that for any function $f(\theta, \phi)$ which can be expanded in a Taylor Series in ϕ with θ -dependent coefficients,

$$e^{-iL_z\phi_0/\hbar}f(\theta, \phi) = f(\theta, \phi - \phi_0) .$$

[You will want to use $L_z = -i\hbar\partial/\partial\phi$.]

Thus, the operator $e^{-iL_z\phi_0/\hbar}$ is an example of a rotation operator, and L_z is called the generator of rotations about the z -axis.