

# Quantum Physics II (8.05) Fall 2004

## Assignment 1

Massachusetts Institute of Technology  
Physics Department  
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*Due September 16, 2004*

This week is dedicated primarily to a quick review of wave mechanics (8.04), from which we will abstract the general principles of quantum mechanics.

### Reading Assignment for the first two weeks of the course

- Review of wave mechanics: **Griffith's Ch. 1 and 2**. Some of the mathematical tools in Ch. 2 are important in 8.05, especially: the Kronecker  $\delta_{ij}$ , the Dirac  $\delta$ -function,  $\delta(x - y)$ , and the Fourier transform.
- For an introduction to linear algebra see: **Griffiths Appendix** and more extensively **Shankar Ch. 1**. We will use the following concepts **extensively** in 8.05: vector spaces, norms, inner products, bases, orthonormality and completeness, eigenvalues, eigenvectors, and eigenfunctions, and somewhat later, linear transformations.
- To read about the Stern-Gerlach experiment, see the handout from **Sakurai Ch. 1**

## Problem Set 1

These problems are intended to help you review and generalize major results from last semester. They cover only the material from the first week's lectures.

### 1. Average momentum [8 points]

- (a) A particle's coordinate space wavefunction is square-integrable and real up to an arbitrary multiplicative phase:

$$\psi(x) = e^{i\alpha}\phi(x)$$

with  $\alpha$  real and constant and  $\phi(x)$  real. Prove that its average momentum is zero.

- (b) Now suppose that  $\alpha \rightarrow \alpha(x)$  (*ie.*  $\alpha$  varies with position), but is still real. What is the average value of the momentum?

### 2. Properties of a wavefunction [12 points]

A particle of mass  $m$  moves on the line  $x \in [-\infty, \infty]$ , and has the following wave function at some time:

$$\begin{aligned} \psi(x) &= N \cos(bx) \quad \text{for } |x| < \frac{\pi}{2b} \\ &= 0 \quad \text{for } |x| > \frac{\pi}{2b} \end{aligned} \quad (1)$$

- (a) Normalize  $\psi(x)$ . That is, find the value of  $N$ .
- (b) What is  $\langle x \rangle$ ? What is  $\langle x^2 \rangle$ ? Is  $\psi(x)$  a position eigenstate?
- (c) What is  $\langle p \rangle$ ? What is  $\langle p^2 \rangle$ ? Is  $\psi(x)$  a momentum eigenstate?
- (d) If the momentum of  $\psi$  is measured, what is the probability distribution,  $\mathcal{P}(p)$ , of the results? Sketch  $\mathcal{P}(p)$ .
- (e) Suppose  $\psi(x)$  describes a free particle, so  $V(x) = 0$ . Is  $\psi(x)$  an energy eigenstate?

### 3. Properties of a another wavefunction (14 points)

A particle of mass  $m$  moving under the influence of a one-dimensional potential  $V(x)$  has the wave function:

$$\psi(x) = Nx \exp(-\alpha x^2) \quad (2)$$

- (a) Normalize  $\psi(x)$ . Assume that  $\alpha > 0$ .
- (b) Is  $\psi(x)$  a position eigenstate? Is  $\psi(x)$  a momentum eigenstate? Explain your reasoning.

- (c) Suppose that I tell you that  $V(x) = 0$ . What is  $\langle \hat{H} \rangle$ ?
- (d) Now, suppose that I do not tell you anything about  $V(x)$ , but tell you instead that  $\psi(x)$  **is** an energy eigenstate. Find the potential  $V(x)$  and the energy eigenvalue  $E$ . If I add a constant to whatever potential you find, my potential will also be acceptable. To remove this ambiguity, let us agree that  $V(0) = 0$ . [Hint: If  $\psi(x)$  is an energy eigenstate, it must obey  $-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$  for some  $V(x)$  and some  $E$ .]
- (e) Using general properties of wave mechanics in one dimension, explain why  $\psi(x)$  is or is not the *ground state* wave function for a particle in the potential  $V(x)$  from (d). (ie. explain why  $E$  is or is not the lowest possible energy eigenvalue.)

Now, instead of (2) consider a new wave function,

$$\psi(x) = Nxe^{-\beta x} \quad (3)$$

where  $\beta > 0$ , and the particle moves on the **half**-line  $x \in [0, \infty]$ , with an impenetrable wall ( $V(x) = \infty$ ) for  $x \leq 0$ , and a potential  $V(x)$  for  $x > 0$ .

- (f) Repeat part (d) for the wavefunction in (3). That is, assume it is an energy eigenstate and find the potential  $V(x)$  and the energy eigenvalue  $E$ . [This time, it is more convenient if we all agree that  $V(\infty) = 0$ .]
- (g) Repeat part (e) for this wave function on the half-line.

#### 4. Conserved probability current [10 points]

Suppose  $\psi(x, t)$  obeys the one-dimensional Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x, t) + V(x)\psi(x, t) = i\hbar\frac{\partial}{\partial t}\psi(x, t). \quad (4)$$

- (a) Derive the conservation law for probability,

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0, \quad (5)$$

where  $\rho(x, t) = \psi^*\psi$  is the probability density and  $J(x, t) = \frac{\hbar}{m} \text{Im} \left( \psi^* \frac{\partial \psi}{\partial x} \right)$  is the probability flux.

- (b) What are the units of  $\rho$  and  $J$ ?
- (c) Explain why (5) is a conservation law for probability. Use it to show that a wavefunction  $\psi(x, t)$  that is normalized at time  $t$  remains normalized at later times.
- (d) Compute the probability flux for  $\psi(x)$  from problem 1(b). Consider the ratio  $J(x)/\rho(x)$  and explain why  $\hbar\alpha'(x)/m$  is used as a measure of the local velocity of the quantum particle described by  $\psi(x) = e^{i\alpha(x)}\phi(x)$ .

5. **An unstable particle**<sup>1</sup> [8 points]

Suppose you want to describe an unstable particle that spontaneously disintegrates with a “lifetime”  $\tau$ . In that case, the total probability for finding the particle should not be constant, but should decrease at an exponential rate,

$$P(t) = \int_{-\infty}^{\infty} dx |\psi(x, t)|^2 = e^{-t/\tau}. \quad (6)$$

Show that this can be incorporated in the Schrödinger equation if we add a *negative imaginary constant* to the potential,

$$V(x) \rightarrow V(x) - \frac{i\Gamma}{2}. \quad (7)$$

with  $\Gamma > 0$ .

(a) Derive the generalization of (5).

(b) Show that  $P(t)$  behaves as given by (6) and relate the lifetime  $\tau$  to  $\Gamma$ .

6. **Sequential Measurements** (8 points)

An operator  $\hat{A}$ , corresponding to an observable  $A$ , has two orthonormal eigenstates  $\psi_1$  and  $\psi_2$ , with eigenvalues  $a_1$  and  $a_2$ . An operator  $\hat{B}$ , corresponding to an observable  $B$ , has two orthonormal eigenstates  $\phi_1$  and  $\phi_2$ , with eigenvalues  $b_1$  and  $b_2$ . The eigenstates are related by

$$\psi_1 = \frac{1}{\sqrt{5}}(\phi_1 + 2\phi_2) \quad \psi_2 = \frac{1}{\sqrt{5}}(2\phi_1 - \phi_2). \quad (8)$$

(a)  $B$  is measured, and the value  $b_1$  is obtained. What is the state of the system after this measurement?

(b) If  $A$  is then measured, and then  $B$  is measured again, what is the probability that the value  $b_1$  is obtained the second time  $B$  is measured?

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<sup>1</sup>Adapted from Griffiths, 1.15