

Quantum Physics II (8.05) Fall 2004

Assignment 4

Massachusetts Institute of Technology
Physics Department
September 30, 2004

*Due October 7, 2004
7:00pm*

This week: unitary time evolution in quantum mechanics and the Schrödinger and Heisenberg pictures. Application to coherent states of the harmonic oscillator. Discussion of the classical limit of quantum mechanics.

Reading Assignment for week four of the course

- Quantum dynamics. Schrödinger and Heisenberg pictures. These subjects are not covered explicitly in Griffiths. Your class notes will contain all the essentials. You can find brief discussions on the two pictures in **Cohen-Tannoudji, Complement G_{III}** or **Schiff, pages 168-171**. (Additional reading on quantum dynamics can be found in Ohanian Ch 5, Cohen-Tannoudji III D, or Sakurai Ch 2.1 & 2.2.)

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Problem Set 4

1. Anharmonic Oscillator [10 points]

Suppose the spring in a harmonic oscillator stiffens slightly with extension. This can be modelled by altering the potential such that:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\hat{x}^4 \quad (1)$$

where λ is a small positive constant.

- Write the *classical* equations of motion for this oscillator (Newton's Law) and show that the added term does what it is supposed to. Why not use \hat{x}^3 ?
- Suppose one prepared the system in a state which is the ground state of the *unperturbed* Hamiltonian (ie the Hamiltonian with $\lambda = 0$). Call this state $|0\rangle$ as usual. What is the expectation value of the complete Hamiltonian given by (1) in the state $|0\rangle$? In other words, how much is the energy of the old ground state shifted by the anharmonic term?
- Show that the old ground state, $|0\rangle$, is not an eigenstate of the anharmonic oscillator for $\lambda \neq 0$.

2. Asymmetric Two Dimensional Oscillator [15 points]

Suppose a particle of mass m is free to move in the (x, y) plane subject to a harmonic potential centered at the origin. But suppose the restoring force in the x and y directions are different. The Hamiltonian for this system is

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}\hat{p}_y^2 + \frac{1}{2}m\omega_x^2\hat{x}^2 + \frac{1}{2}m\omega_y^2\hat{y}^2 \quad (2)$$

where $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar$, and all other commutators between \hat{x} , \hat{y} , \hat{p}_x and \hat{p}_y are zero.

- Introduce lowering and raising operators \hat{a}_x , \hat{a}_y , \hat{a}_x^\dagger and \hat{a}_y^\dagger as well as $\hat{N}_x = \hat{a}_x^\dagger\hat{a}_x$ and $\hat{N}_y = \hat{a}_y^\dagger\hat{a}_y$. What is \hat{H} in terms of these operators? Find expressions for the energy eigenstates and the energy eigenvalues.
[The analogous results for the one-dimensional oscillator were $|n\rangle = \frac{1}{\sqrt{n!}}[\hat{a}^\dagger]^n|0\rangle$ and $E_n = \hbar\omega(n + \frac{1}{2})$. Here, you will want to define an n_x and n_y .]

- Plot an energy level diagram for this system. Let's assume, just to clarify the pictures, that $\omega_x \approx \omega_y$, and to be definite take $\omega_x > \omega_y$. Include at least the first three groups of states. Mark their values of n_x and n_y .

Now define new operators,

$$\begin{aligned} \hat{N} &= \hat{N}_x + \hat{N}_y \\ \hat{n} &= \hat{N}_x - \hat{N}_y \end{aligned}$$

and notice that they commute with \hat{H} . The energy eigenstates can therefore be labelled by n and N , the eigenvalues of \hat{n} and \hat{N} . What is $E_{N,n}$? Redraw the energy level diagram and label the states with the quantum numbers n and N .

- (c) Use your pictures to decide which of the following are complete sets of commuting observables: $\{\hat{N}\}$, $\{\hat{N}, \hat{n}\}$, $\{\hat{N}_x, \hat{N}_y\}$, and $\{\hat{H}\}$. How do your answers change if you take $\omega_x = \omega_y$? How do your answers change if ω_x/ω_y is equal to a rational number?
- (d) Now let $\omega_x = \omega_y = \omega$, and define the angular momentum operator

$$\hat{\ell} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x . \quad (3)$$

Write $\hat{\ell}$ in terms of the operators \hat{a}_x , \hat{a}_y , \hat{a}_x^\dagger and \hat{a}_y^\dagger . Show that $\hat{\ell}$ commutes with \hat{H} and that therefore both can be simultaneously diagonalized.

- (e) **This last part of this problem will not be graded.** We will return to the following question later in the course, but I pose it here for you to think about and solve now if you choose. Continuing from part (d), consider the degenerate subspace consisting of all the energy eigenstates which have the N^{th} energy eigenvalue. Find a basis for this subspace such that the basis vectors are eigenstates of $\hat{\ell}$. Classify these basis states by their angular momentum eigenvalues, and show that \hat{H} and $\hat{\ell}$ together constitute a complete set of commuting observables for the entire Hilbert space.

3. Useful Operator Identities & Translations [15 points]

Because operators do not commute, it requires care to evaluate some apparently simple expressions. Some particular expressions involving exponentials of operators are very useful in quantum mechanics. We first encounter them in the study of coherent states of the harmonic oscillator, but we will use them again. This problem leads you through the derivations, and some simple applications.

Suppose that \hat{A} and \hat{B} are two operators that do not commute, $[\hat{A}, \hat{B}] \neq 0$.

- (a) Let z be a complex variable and show that

$$\frac{d}{dz} e^{z(\hat{A}+\hat{B})} = (\hat{A} + \hat{B}) e^{z(\hat{A}+\hat{B})} = e^{z(\hat{A}+\hat{B})} (\hat{A} + \hat{B}) .$$

- (b) Now suppose $[\hat{A}, \hat{B}] = c$, where c is a c -number (a complex number times the identity operator). Prove that

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + c . \quad (4)$$

[Hint: Define an operator-valued function $\hat{F}(z) \equiv e^{z\hat{A}} \hat{B} e^{-z\hat{A}}$. What is $\hat{F}(0)$? Derive a differential equation for $\hat{F}(z)$ by differentiating with respect to z . Integrate the equation.]

- (c) Let a be a real number and \hat{p} be the momentum operator. Show that the unitary **translation operator**

$$\hat{T}(a) \equiv e^{-ia\hat{p}/\hbar}$$

translates the position operator:

$$\hat{T}^\dagger(a) \hat{x} \hat{T}(a) = \hat{x} + a .$$

- (d) Demonstrate that if a state $|\psi\rangle$ is described by the wave function $\langle x|\psi\rangle = \psi(x)$, then the state $\hat{T}(a)|\psi\rangle$ is described by the wave function $\psi(x - a)$.

[Note: Recall $\hat{T}(a)$ was built out of the momentum operator. Due to the properties of $\hat{T}(a)$ we call \hat{p} the “generator” of translations.]

- (e) Again, suppose $[\hat{A}, \hat{B}] = c$, where c is a c-number. Prove

$$e^{\hat{A}+\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{c/2} = e^{\hat{A}} e^{\hat{B}} e^{-c/2} .$$

(This is a special case of the Baker-Hausdorff theorem, which you might want to look up in a graduate text on quantum mechanics.) [Hint: Try starting with $\hat{G}(z) \equiv e^{z(\hat{A}+\hat{B})} e^{-z\hat{A}}$. Consider $\hat{G}^{-1} \frac{d}{dz} \hat{G}(z)$.]

4. Time Evolution in the Heisenberg Picture [15 points]

In this problem we'll study the time evolution of a wave packet acted upon by a constant force. This is a case where the Schrödinger equation is hard to solve, but the Heisenberg equations of motion for the time dependence of operators can be solved easily and quite a bit can be learned about the motion.

Suppose a quantum particle is described the Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + g\hat{x}$$

We all know that this corresponds to the particle subject to a constant force, $F = -dV/dx = -g$.

- (a) Use the Heisenberg equations of motion,

$$i\hbar \frac{d\hat{A}_H}{dt} = [\hat{A}_H, \hat{H}]$$

to show that the Heisenberg operators, $\hat{x}_H(t)$ and $\hat{p}_H(t)$ obey an analog of Newton's law, $F = ma$.

- (b) Integrate the Heisenberg equations of motion to obtain $\hat{x}_H(t)$ in terms of $\hat{x}_H(0)$ and $\hat{p}_H(0)$.

(c) Suppose that at $t = 0$ a particle has coordinate space wavefunction,

$$\langle x|\psi\rangle = \psi(x) = Ne^{-\frac{x^2}{\Delta^2}},$$

where N is a constant that normalizes ψ to unity. Compute $\langle\psi|\hat{x}_H(t)|\psi\rangle$ and show that it behaves classically.

(d) Now compute the $(\Delta x(t))^2$, the squared uncertainty in x ,

$$(\Delta x(t))^2 = \langle\hat{x}_H^2(t)\rangle - \langle\hat{x}_H(t)\rangle^2$$

and $\langle\hat{A}_H\rangle \equiv \langle\psi|\hat{A}_H|\psi\rangle$.

Show that $(\Delta x(t))^2$ grows quadratically with time,

$$(\Delta x(t))^2 = (\Delta x(0))^2 + \lambda t^2$$

and find the coefficient λ . How does the spreading of the wavepacket depend on the value of g ?

5. Uncertainty Principle for Spin [5 points]

In lecture we derived the generalized uncertainty principle

$$(\Delta A)^2 (\Delta B)^2 \geq \left\langle \frac{1}{2i} [\hat{A}, \hat{B}] \right\rangle^2. \quad (5)$$

Recall from problem set 3 that the spin operators $\hat{S}_i = \frac{\hbar}{2}\sigma_i$ where σ_i are the Pauli matrices. Consider the state $|z; +\rangle = |+\rangle$ and verify that (5) is satisfied for the operators $\hat{A} = \hat{S}_x$, $\hat{B} = \hat{S}_y$.