Quantum Physics II (8.05) Fall 2004
Assignment 6

Massachusetts Institute of Technology
Physics Department *Due October 21, 2004*
October 14, 2004 *7:00pm*

This week we continue our discussion of the quantum mechanics of two state systems. Spin in a magnetic field, eigenstates, spin precession and nuclear magnetic resonance. After that we will discuss neutrino oscillations and kaon phenomena.

Reading Assignment for week six of the course

The reading assignment for this week is a continuation of last week's, with the addition of the recommended reading from Feynman on the ammonia molecule and the course notes on neutrino oscillations. These readings will cover the material up until the midterm.

- (continued from last week) A detailed discussion of two state systems can be found in Cohen-Tannoudji. Complement F_{IV} is on nuclear magnetic resonance, and is **recommended as an addition to the lecture material on NMR.**

- Recommended: Feynman, Vol. III, §8.6 The Ammonia Molecule (and also §9 The Ammonia Maser, although we will not cover this material in lecture).

Problem Set 6

1. **Spin one half particle quantized along an arbitrary axis** [18 points]

When discussing spin, you might wonder, "What's so special about the z-axis? Or for that matter the x- or y-axes?" This problem will show you that the answer is "Nothing."

Define a spin operator $\hat{S}_{\vec{n}}$ as follows:

$$\hat{S}_{\vec{n}} = \vec{n} \cdot \vec{\hat{S}} = n_x \hat{S}_x + n_y \hat{S}_y + n_z \hat{S}_z \ . \tag{1}$$

Here $\vec{n} = (n_x, n_y, n_z)$ is a unit vector (of real numbers) and $\vec{\hat{S}}$ is a vector of operators. (Recall that the spin operators are defined by $\hat{S}_x = \frac{\hbar}{2}\sigma_1$, $\hat{S}_y = \frac{\hbar}{2}\sigma_2$ and $\hat{S}_z = \frac{\hbar}{2}\sigma_3$, where σ_i are the now familiar Pauli matrices. Note that the Pauli matrices are sometimes referred to as $\sigma_{x,y,z}$ and sometimes as $\sigma_{1,2,3}$.)

(a) Define the angles ϕ and θ by

$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \ .$$

Write the operator $\hat{S}_{\vec{n}}$ in terms of the eigenkets of $\hat{S}_z$, namely $|+\rangle$ and $|-\rangle$, the corresponding bras, and the angles ϕ and θ. Now write the operator $\hat{S}_{\vec{n}}$ as a matrix, using the eigenkets of $\hat{S}_z$ as the basis vectors as usual.

(b) Calculate the eigenvalues of $\hat{S}_{\vec{n}}$, and show that they are the same as the eigenvalues of $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$.

(c) Construct the corresponding normalized eigenstates, which we will write as $|\vec{n};+\rangle$ and $|\vec{n};-\rangle$, which satisfy

$$\hat{S}_{\vec{n}}|\vec{n};\pm\rangle = \pm\frac{\hbar}{2}|\vec{n};\pm\rangle \ .$$

Note: The answer is

$$\begin{aligned} |\vec{n};+\rangle &= \cos(\theta/2)|+\rangle + \sin(\theta/2)\exp(i\phi)|-\rangle \\ |\vec{n};-\rangle &= \sin(\theta/2)|+\rangle - \cos(\theta/2)\exp(i\phi)|-\rangle \end{aligned} \tag{2}$$

but do not just verify that this satisfies the above eigenvector equation. Rather, treat the problem as a straightforward eigenvector problem. You may find eigenvectors which differ from those in (2). If so, verify that yours are physically equivalent.

(d) Check that (2) are orthonormal and that they reduce to eigenvectors for $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ when $\vec{n}$ points in the x, y, and z directions.

(e) Suppose you have just made a measurement with a suitably oriented Stern-Gerlach experiment, and so know that you have a spin-1/2 atom in the state $|\vec{n};+\rangle$, where $\vec{n}$ lies in the (x,z)-plane. (That is, $\phi = 0$; $\theta \neq 0$.) If you now pass this atom through a second Stern-Gerlach magnet, designed to measure the x-component of the spin, what is the probability that you will find the x-component of the spin to be $+\hbar/2$?

(f) Consider the operator $\hat{R}_y(\theta) \equiv \exp\left(-\frac{i\theta\hat{S}_y}{\hbar}\right)$ where θ is a real number. Express $\hat{R}_y(\theta)$ as a linear combination of the unit operator I and $\hat{S}_y$. Show that $\hat{R}_y(\theta)^\dagger = \hat{R}_y(-\theta)$. What is the operator $\hat{R}_y(\theta)\hat{S}_z\hat{R}_y(\theta)^\dagger$? [Results from your last problem set will be useful. Do clearly state which previously obtained result(s) you are using.]

(g) Find the states which are obtained if one acts with the operator $\hat{R}_y(\theta)$ on the states $|+\rangle$ and $|-\rangle$. For what operator are the states that you found eigenstates? Explain why we can think of $\hat{R}_y(\theta)$ as a rotation operator.

2. **Time Evolution in a Two-State Problem** [10 points]

This problem is a slight modification of Sakurai's 2.9.

A box containing a particle is divided into a right and left compartment by a partition. If the particle is known to be on the left side with certainty, we call the state $|L\rangle$; if on the right, we call the state $|R\rangle$. (Of all the different possible $|L\rangle$ states, corresponding to different spatial shapes of the wave function within the left half of the box, we consider only the one with the lowest energy. Same on the right. Hence, we have a two-state problem.) Assume that the box is symmetric, in the sense that $\langle L|\hat{H}|L\rangle = \langle R|\hat{H}|R\rangle$. Let us now shift our zero of energy in such a way that $\langle L|\hat{H}|L\rangle = \langle R|\hat{H}|R\rangle = 0$. The Hamiltonian does not vanish, however, because the particle can tunnel through the partition. This tunnelling effect is characterized by the Hamiltonian

$$\hat{H} = \Delta(\,|L\rangle\langle R| + |R\rangle\langle L|\,)$$

where Δ is a real number with the dimension of energy.

(a) Find the normalized energy eigenstates, and the corresponding energy eigenvalues.

(b) In the Schrödinger picture, the basis kets $|L\rangle$ and $|R\rangle$ are fixed while the state vector moves with time. Suppose that at time $t = 0$ the state vector is given by

$$|\alpha(0)\rangle = c_L|L\rangle + c_R|R\rangle \ .$$

Find $|\alpha(t)\rangle$, the state vector at a later time, by applying the time evolution operator.

(c) Suppose that at time $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?

(d) Suppose that I had made an error and wrote $\hat{H}$ as

$$\hat{H} = \Delta |L\rangle\langle R| \ .$$

By explicitly solving the most general time evolution problem with this Hamiltonian (as you did in part (b) for the correct Hamiltonian) show that probability conservation is violated.

3. **Time Evolution in a Three-State System** [15 points]

Carbon dioxide is a linear molecule (OCO) which can pick up an extra electron and become a negatively charged ion. Suppose that the electron would have energy E_O if it were attached to either oxygen atom, or energy E_C if it were attached to the carbon atom in the middle. Call these states $|L\rangle$, $|C\rangle$ and $|R\rangle$, for left oxygen, carbon, and right oxygen. The energy eigenstates need not, however, have either energy E_O or E_C because there is some probability that the electron may hop between an oxygen atom and the carbon atom. (Assume that the probability of jumping directly from oxygen to oxygen can be neglected.)

(a) Write down a model Hamiltonian to describe the physics of the electron outlined above. You will need to introduce a new parameter. Find the energy eigenvalues of this three-state system in terms of E_C, E_O, and your new parameter.

(b) In this part and the next, assume $E_C = E_O$. Find the energy eigenstates.

(c) Assume that at time $t = 0$ the electron is in state $|L\rangle$. That is, it is localized on the left oxygen atom. This is not a stationary state. What is the probability that at some later time t, the electron will be in state $|L\rangle$? In state $|C\rangle$? In state $|R\rangle$? Plot these three probabilities as functions of time.

4. **Precession in a Magnetic Field** [10 points]

An electron sits in a constant 10 Kilogauss magnetic field along the $\hat{z}$ axis.

(a) What is the spin precession frequency? Don't just give a formula, also plug in numbers.

(b) If the initial state of the electron is $|\psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ (where $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of $\hat{S}_z$) what is the expectation value of the (vector) spin operator $\vec{S}$ as a function of time?

You might find these units unfamiliar. The Earth's magnetic field at the surface of the Earth is about half a Gauss. Magnetic fields used in particle detectors range up to 10 Tesla (1 Tesla = 10 kilogauss). Magnetic fields are measured in Gauss in "Gaussian units", which are the cgs units generalized for E&M. Here are a couple of conversion factors that will make it possible to use these units without previous experience: $\mu_{\rm Bohr} = \frac{e\hbar}{2m_e c} = 5.788 \times 10^{-9}$ eV/Gauss; $\hbar c = 197.3 \times 10^{-9}$ eV·m.

5. **The Basics of NMR** [7 points]

   The intention of this problem is to have you work through the steps in the derivation of the behavior of a magnetic moment in a time dependent magnetic field, which you will have seen in lecture.

   An atom with spin $\frac{1}{2}$ and magnetic moment $\vec{\mu} = \gamma \vec{S}$ sits in a time dependent magnetic field,

   $$\vec{B}(t) = B_0 \vec{e}_3 + B_1(\vec{e}_1 \cos \omega t - \vec{e}_2 \sin \omega t) \,,$$

   where the $\vec{e}_i$'s are the usual orthogonal unit vectors.

   (a) First suppose $B_1 = 0$. What is the precession frequency of the magnetic moment around the $\vec{e}_3$ axis?

   (b) Now take $B_1 \neq 0$. Show that the Hamiltonian for this problem can be written

   $$H = \mathcal{U}(t) \; \Theta \; \mathcal{U}^\dagger(t)$$

   where $\mathcal{U}(t)$ is a time dependent unitary operator and Θ is a time-independent operator depending on $\vec{S}$. Find $\mathcal{U}$ and Θ. (You can use results derived in previous problem sets.)

   (c) Let $B_0 \gg B_1$. What is the condition on ω for resonance flipping of the nuclear spin? What is the frequency at which it flips?