# Quantum Physics II (8.05) Fall 2004
# Assignment 8

Massachusetts Institute of Technology
Physics Department
October 28, 2004

*Due November 4, 2004*
*7pm*

## Week 8

Kaon physics and then on to quantum mechanics in three dimensions, initially focusing on angular momentum.

## Reading Assignment for week 8 of the course

For kaon physics the reading material is from the course notes. The material on quantum physics in three dimensions can be found in Griffiths and other standard texts.

- Griffiths §4.1.1 on separation of variables.
- Griffiths §4.3 on angular momentum.

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# Problem Set 8

1. **Kaon Correlations in the Decay of the $f(1270)$ Meson** [10 POINTS]

   In lecture we discussed the correlations an experimenter can observe in the decay of the $\phi$-meson. Here you are asked to work out the same type of correlation information when the initial state differs by a relative sign.

   The $f(1270)$ meson is a quark-antiquark bound state that decays in many different ways. About 2.3% of the time it decays into $K^0\overline{K}^0$. In the $f$ center of mass frame, the kaons come off back-to-back. At the moment of decay, the state of the $K^0\overline{K}^0$ system is

$$|K\overline{K}\rangle = \frac{1}{\sqrt{2}}\Big\{|K^0(\vec{p})\overline{K}^0(-\vec{p})\rangle + |\overline{K}^0(\vec{p})K^0(-\vec{p})\rangle\Big\}\ .$$

   Note the relative + sign.*

   (a) Rewrite this state in the $K_L$, $K_S$ basis.

   (b) Suppose an experiment is designed to detect the kaons by the decays $K_L \to 3\pi$ and $K_S \to 2\pi$. Describe the correlated observations which experimenters on opposite sides of the $f$-decay would make.

2. **An unusual result?** [15 POINTS]

   Let's review some of the properties of the strong interactions that produce kaons. The $K^0$ has strangeness $S = -1$ and the $\bar{K}^0$ has strangeness $S = +1$. The strong interactions conserve strangeness, so when kaons are created (without any other strange particles) they have to be created in $K\bar{K}$ pairs. In particular, in electron-positron annihilation we expect $e^+e^- \to K\bar{K}$. Of course, experimenters measure $K_S$ and $K_L$, not $K^0$ and $\bar{K}^0$.

   Suppose an experimenter report the results of an experiment of the form $e^+e^- \to$ two kaons. The $e^+$ and $e^-$ energies are equal and remain fixed throughout the experiment. The beam energies have a certain spread, usually less than 1% of the energy itself. The experimenter observes $K_L$ or $K_S$ coming out on the near ($n$) or far ($f$) side of the apparatus. After years of patient data taking, he claims that 50% of the time he sees a $K_S$ on both the near and far side , 25% of the time he sees a $K_S$ on the near side and a $K_L$ on the far side, and 25% of the time vice versa. Your job is to see if this experimenter's claims are legitimate or erroneous.

   (a) Can this data be fit by assuming the production of a single meson $|\Psi\rangle$ (like the $f_0(1270)$), which subsequently decays into two kaons?

   (b) Another possibility is that the experiment is actually producing *two* different mesons with almost the same mass. Can the data be fit by assuming one meson is decaying into $K_LK_S$ and the other is decaying into $K_SK_S$?

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*Aside: the sign is in fact determined by the total angular momentum, including orbital and spin, of the initial meson. The $\phi$ has total angular momentum 1; the $f$ has total angular momentum 2.

3. $K_L - K_S$ **Mass Difference** [25 POINTS]

The fact that the $K_L$ and $K_S$ have different decays generates a tiny mass difference between the two states. This problem explores the associated phenomena.

The supplementary notes on kaon physics declare (between eqs. (10) and (11)) that the masses of the $K^0$ and $\overline{K}^0$ are equal because of a deep symmetry of Nature. A more precise statement would be summarized by the following equations:

$$\begin{aligned} \hat{H}|K^0\rangle &= Mc^2|K^0\rangle + \frac{\delta Mc^2}{2}|\overline{K}^0\rangle \\ \hat{H}|\overline{K}^0\rangle &= Mc^2|\overline{K}^0\rangle + \frac{\delta M^* c^2}{2}|K^0\rangle \ , \end{aligned} \tag{1}$$

where $\hat{H}$ is the total Hamiltonian including the effects of both strong and weak interactions. So the diagonal elements of $\hat{H}$ are equal (that's the deep symmetry), but the off-diagonal elements are arbitrary so long as $\hat{H}$ is Hermitian. We will take $\delta M = \delta M^*$. [In fact, $\delta M$ actually has a small imaginary part. This signifies that time reversal invariance is not a perfect symmetry in Nature. We will not discuss this effect here. After all, we should leave some of the surprises which the kaons have in store for you for later in your education!]

- Show that the eigenstates of $\hat{H}$ are $|K_L\rangle$ and $|K_S\rangle$ with a mass difference $|M_L - M_S| = |\delta M|$.

This confirms what you should already have known from our analyses of other two-state systems: if $|K^0\rangle$ and $|\overline{K}^0\rangle$ are *not* eigenstates of the Hamiltonian, and $|K_L\rangle$ and $|K_S\rangle$ *are*, then $\delta M$ defined above must be nonzero, although it can be arbitrarily small. The experimental value is in fact very small:

$$\frac{M_L - M_S}{\overline{M}} = 7.014 \pm 0.018 \times 10^{-15} \ , \tag{2}$$

where $\overline{M}$ is the average of the $K_L$ and $K_S$ masses. ($\overline{M}c^2 = 497.672 \pm 0.031$ MeV.) How could such a tiny mass difference be observed? The answer, of course, lies in the unusual quantum physics of the neutral kaons. Here, the key is the study of the time dependence of the $K^0$ and $\overline{K}^0$ content of a kaon beam.

It is necessary to remember that a particle can be identified as a $K^0$ or $\overline{K}^0$ when it is created or when it interacts with matter. On the other hand, the Hamiltonian eigenstates are $|K_L\rangle$ and $|K_S\rangle$ and

$$\begin{aligned} \hat{H}|K_L\rangle &= (M_L c^2 - i\hbar/2\tau_L)|K_L\rangle \\ \hat{H}|K_S\rangle &= (M_S c^2 - i\hbar/2\tau_S)|K_S\rangle \end{aligned} \tag{3}$$

where we have included the imaginary terms that account for the decay of the $|K_L\rangle$ and $|K_S\rangle$.

Suppose a reaction creates a $K^0$ at time $t = 0$. Let us sit in its rest frame and study its time evolution. Call this state $|\psi(t)\rangle$, with $|\psi(0)\rangle = |K^0\rangle$.

(a) What is $|\psi(t)\rangle$ in the $|K_L\rangle$, $|K_S\rangle$ basis?

(b) What is $|\psi(t)\rangle$ in the $|K^0\rangle$, $|\overline{K}^0\rangle$ basis?

(c) Evaluate the probability to observe a $\overline{K}^0$ (known as the $\overline{K}^0$ "intensity" $I(\overline{K}^0)$ as a function of time?

Your answer should be:

$$I(\overline{K}^0) = \frac{1}{4}\left\{ e^{-\frac{t}{\tau_S}} + e^{-\frac{t}{\tau_L}} - 2e^{-\frac{t}{2}\left(\frac{1}{\tau_L}+\frac{1}{\tau_S}\right)} \cos\left[(M_L - M_S)c^2 t/\hbar\right] \right\} \tag{4}$$

(d) Use the following values

$$\begin{aligned} \tau_S &= 0.8926 \pm 0.0012 \times 10^{-10} \text{ sec} \\ \tau_L &= 5.17 \pm 0.04 \times 10^{-8} \text{ sec} \\ \frac{(M_L - M_S)c^2}{\hbar} &= 0.5333 \pm 0.0027 \times 10^{10} \text{ sec}^{-1} \end{aligned}$$

to plot the $\overline{K}^0$ intensity as a function of time for $M_L = M_S$, and for $M_L - M_S$ as in Nature. It is easiest to measure time in units of $\tau_S$. Discuss the effect of the $K_L - K_S$ mass difference.

(e) You have now determined the time dependence of the $\overline{K}^0$ intensity in the particle's rest frame. Explain (in words) how you might go about mounting an experiment to measure the $K_L - K_S$ mass difference. To help you: A typical reaction that can be initiated by $\overline{K}^0$ but not by $K^0$ is $\overline{K}^0 + p \rightarrow \pi^+ + \Lambda^0$, where the $\Lambda^0$ is a particle (a bound state of one u, one d, and one s quark) which lives long enough to make it relatively easy to detect. You may assume that it is possible to prepare a relatively pure and relatively mono-energetic beam of $K^0$'s. You ought to consider over what distance or time scales you want to probe $I(\overline{K}^0)$. How far can a kaon you have made travel in $\tau_S$? Can you use relativity to help you in this experiment? Suppose that you had a choice of proposing this experiment at Brookhaven National Lab, where the kaons would have kinetic energies of about 10 GeV, or at Fermilab, where they would have kinetic energies of about 200 GeV. Where would you propose to do the experiment?

4. **Properties of the Angular Momentum Operators** [10 points]
Like the raising and lowering operators for the harmonic oscillator, the components of the angular momentum have simple commutation relations that will help us understand the physics of angular motion. Note how similar their properties are to the $\sigma$ matrices. They are both deeply connected with the theory of rotations in space. **Note that $\vec{L}$, $\vec{x}$, and $\vec{p}$ are all operators in this problem. So for simplicity, we drop the "hat", $\hat{\mathcal{O}}$, notation for operators.** Define the Cartesian components of the angular momentum operator in the usual way:

$$\begin{aligned} \vec{L} &= \vec{x} \times \vec{p} \\ L_j &= \varepsilon_{jkm} x_k p_m . \end{aligned} \tag{5}$$

[Repeated indices are summed.] The commutators of the angular momentum operators are determined by the basic canonical commutators,

$$\begin{aligned} [x_j, p_k] &= i\hbar\delta_{jk} \\ [x_j, x_k] &= 0 \\ [p_j, p_k] &= 0 \end{aligned} \tag{6}$$

(a) Prove $[L_j, L_k] = i\hbar\varepsilon_{jkm} L_m$. [Hint: use properties of the product of two $\varepsilon$-symbols derived in previous problem sets. Don't do every single choice of the indices $j$ and $k$ as a separate calculation!]

(b) Prove $[\vec{L}^2, L_k] = 0$, where $\vec{L}^2 = L_1^2 + L_2^2 + L_3^2$.
[Thus, one can simultaneously use the total angular momentum $l$ associated with $L^2$ and one component $m$ (typically the $z$ component) to classify states as was done with the Hydrogen atom wavefunctions.]

(c) Define $L_\pm = L_1 \pm iL_2$. Calculate $[L_\pm, L_3]$ and $[L_+, L_-]$.
[Using these relations we can show that if $|m\rangle$ is an eigenstate of $L_3$: $L_3|m\rangle = m|m\rangle$, then $L_\pm|m\rangle$ are also eigenstates. The operators $L_\pm$ allow us to construct and classify the eigenstates of angular momentum much as $a$ and $a^\dagger$ allowed us to construct and classify the eigenstates of the harmonic oscillator from $|0\rangle$.]