Massachusetts Institute of Technology
Department of Physics

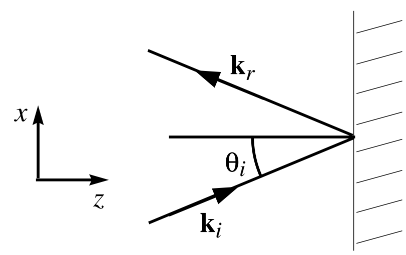
Physics 8.07 Fall 2004

Final Exam

1. Reflection by a Perfect Conductor (25 points)
A monochromatic plane electromagnetic wave of wavevector

$$\vec{k}_i = \frac{\omega}{c}\hat{k}_i \ , \quad \hat{k}_i = \vec{e}_x \sin\theta_i + \vec{e}_z \cos\theta_i \ ,$$

is incident at angle θ_i to the normal to a perfect conductor at $z = 0$, as shown in the diagram below. The conductor covers the entire x-y plane; the region $z < 0$ is vacuum.



The incident electric field is linearly polarized:

$$\vec{E}_i = E_i \cos(\vec{k}_i \cdot \vec{x} - \omega t)\, \vec{n}_i \ ,$$

where $\vec{n}_i$ is a unit vector in the x-z plane.

a) Determine the electric polarization vector $\vec{n}_i$ and the magnetic field $\vec{B}_i$ of the incident plane wave.

b) Write down a trial form for the reflected fields $\vec{E}_r$ and $\vec{B}_r$ using the wavevector $\vec{k}_r = (\omega/c)\hat{k}_r$ and appropriate amplitudes and polarization vectors. Verify that your reflected wave obeys the Maxwell equations for $z < 0$.

c) Apply the boundary condition $[\vec{e}_z \times \vec{E}\,] = 0$ at $z = 0$ to determine $\hat{k}_r$ and exact expressions for $\vec{E}_r$ and $\vec{B}_r$.

d) What are the surface charge density $\sigma(x, t)$ and surface current density $\vec{K}(x, t)$? Show that they obey the equation of charge conservation,

$$\frac{\partial \sigma}{\partial t} + \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} = 0 \ .$$

2. **Cyclotron Radiation (25 points)**

An electron (charge $-e$, mass m) of speed $v \ll c$ moves perpendicularly to a uniform magnetic field $\vec{B} = B\vec{e}_z$.

a) Determine the electron's trajectory $\vec{w}(t) = (x(t), y(t))$ assuming that the time-average position is $x = y = 0$ and the electron has $y = 0$ at $t = 0$.

b) The Liénard-Wiechert potentials are given by the formulae

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \frac{(-e)c\vec{v}}{(rc - \vec{r} \cdot \vec{v})} = \frac{\vec{v}}{c^2} V(\vec{x}, t) \ .$$

Consider an observation point $\vec{x}$. Give complete expressions for $\vec{r}$ and $\vec{v}$ in terms of $(\vec{x}, t)$ and the function $\vec{w}(t)$ (which you may evaluate at any necessary time).

c) Assuming that $r \gg |\vec{w}|$, and approximating the retarded time $t_{\rm ret}$ by its value if the source were at $\vec{w} = 0$, show that $\vec{\nabla} t_{\rm ret} \approx -\hat{r}/c$.

d) Using the results of parts b) and c), show that at sufficiently large distances the electric field is

$$\vec{E} \approx -\frac{\mu_0 e}{4\pi r}[\hat{r} \times (\hat{r} \times \vec{a})] \ .$$

(Hint: expand $\vec{A}$ and V only to first order in v/c.) Evaluate this when the observation point is at a distance r, (i) along the z-axis and (ii) along the x-axis. (Your answer to this part need not distinguish between t and $t_{\rm ret}$.)

e) Determine the time-average Poynting flux of the radiation emitted by the electron, for the two cases in part c).

3. **Dielectric sphere (25 points)**
A dielectric sphere of constant permittivity $\epsilon = \kappa\epsilon_0$ and radius R is placed in an otherwise uniform electric field $\vec{E}_0 = E_0\vec{e}_z$. The field is modified by the presence of the dielectric sphere.

a) Write down the form of the electrostatic potential $V(r, \theta)$ for $r < R$ and for $r > R$, using as many terms of the multipole expansion as are necessary.

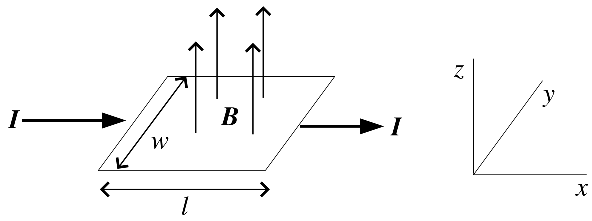
b) Determine $V(r, \theta)$ everywhere. Your results should depend on κ, R, and E_0.

c) Determine $\vec{E}(r, \theta)$ everywhere.

d) Determine the bound surface charge $\sigma_b(\theta)$.

e) What is the total electric dipole moment induced in the dielectric sphere?

Multiple Choice Questions (5 points each, no partial credit)



4. A current I flows along $\vec{e}_x$ through a rectangular plate of conducting material (of width w, length l, and negligible thickness) in the presence of a uniform magnetic field $\vec{B} = B\vec{e}_z$. The charge carriers are electrons which move with speed v in the direction $-\vec{e}_x$. What is the potential difference between the top ($y = w$) and bottom ($y = 0$) of the bar?
   (a) vwB.
   (b) $IwlB/e$.
   (c) $-vwB$.
   (d) $-IwlB/e$.

5. The Maxwell stress tensor of a uniform electric field corresponds to which case?
   (a) Isotropic pressure.
   (b) Isotropic tension (negative pressure).
   (c) Pressure along the field lines and tension perpendicular to them.
   (d) Tension along the field lines and pressure perpendicular to them.

6. Which of the following is most responsible for paramagnetism ($0 < \chi_m \ll 1$)?
   (a) Lenz's law.
   (b) Alignment of permanent dipoles.
   (c) The Pauli exclusion principle.
   (d) Superconductivity.

There are two more problems on the next page.

7. A circular loop of radius r carries current I. What is the magnetic field in the center of the loop?

(a) $\mu_0 I/(2\pi r)$.
(b) $\mu_0 I/(\pi r)$.
(c) $\mu_0 I/(2r)$.
(d) $\mu_0 I/(r)$.

8. A charge $q > 0$ is placed off-center inside a thin spherical conducting shell with zero net charge aside from the point charge q. Circle the case below that shows the correct electric field lines.