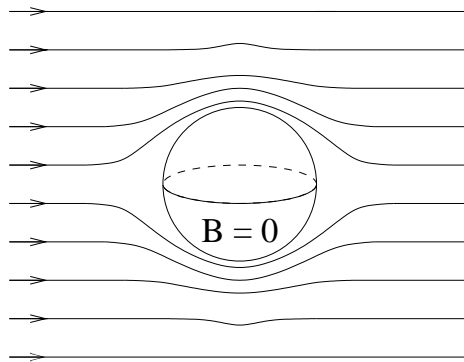


Final Exam

1. Superconductor in a constant magnetic field (25 points)

Superconductors have the interesting property that they exclude magnetic fields from their interiors. This is similar to the way that a conductor excludes electric fields but with a slightly different boundary condition. While conductors exclude electric fields by having a surface charge, superconductors exclude magnetic fields by having a surface current. Boundary conditions on the magnetic field then imply that, just outside the superconductor, $\vec{B} \cdot \vec{n} = 0$ where \vec{n} is a normal to the surface of the superconductor.

Consider putting a spherical superconductor of radius R in a constant magnetic field $\vec{B} = B\vec{e}_z$. The approximate field lines are sketched below:



Your task is to figure out the magnetic field everywhere outside the superconductor. (Hint: this is mathematically similar to an electrostatics problem!)

- a) Since outside the superconductor \vec{B} satisfies $\vec{\nabla} \times \vec{B} = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$, we can write $\vec{B} = \vec{\nabla}U$ where U satisfies Laplace's equation. What are the boundary conditions on U at infinity and on the surface of the sphere?
- b) Expand U in Legendre polynomials (and powers of r where r is the distance from the center of the sphere) for $r > R$. Find the coefficients of the terms in the series.
- c) Using your result from part (b), find \vec{B} everywhere.
- d) From the boundary condition on \vec{B}^{\parallel} , find the surface current (magnitude and direction, as a function of θ) at $r = R$ on the superconductor.

2. Thomson Scattering (25 points)

A plane electromagnetic wave with electric field

$$\vec{E} = E_0 \cos(kz - \omega t) \vec{e}_x$$

is incident upon an electron of charge $-e$ and mass m at $\vec{x} = 0$. The electron is free, i.e. it is not bound inside an atom.

- a) What is the magnetic field of the incident plane wave?
- b) Assuming that the particle moves nonrelativistically and is displaced much less than a wavelength of the incident plane wave, derive the acceleration $\vec{a}(t)$ of the electron.
- c) For a nonrelativistically moving charge q , at large distances the radiative electric field is

$$\vec{E} = \frac{\mu_0 q}{4\pi r} [\hat{r} \times (\hat{r} \times \vec{a})] .$$

Determine the instantaneous Poynting flux of the radiation emitted by the electron, as a function of (r, θ, t) . (Your answer to this part need not distinguish between t and retarded time.) Make a sketch indicating what \hat{r} , r and θ are.

- d) Starting from the Poynting flux, determine the time-average power radiated by the electron, integrated over all solid angle.
- e) The intensity I_0 of the incident plane wave and the power radiated by the charge are related by $P = I_0 \sigma_T$ where σ_T is the Thomson cross section. Find σ_T in terms of the classical electron radius $r_0 = \mu_0 e^2 / (4\pi m)$.

3. Charge Sheet (25 points)

A uniform sheet of charge fills the $z = 0$ plane so that $\rho(\vec{x}) = \sigma\delta(z)$. At time $t = 0$ the sheet begins to move in the y -direction with constant speed v so that for $t > 0$ the current density is $\vec{J}(\vec{x}) = \sigma v\delta(z)\vec{e}_y$ while $\rho(\vec{x}) = \sigma\delta(z)$ is unchanged. In other words, (ρ, \vec{J}) is given for all times by $(\rho, \rho\vec{v})$ where $\vec{v} = 0$ for $t < 0$ and $\vec{v} = v\vec{e}_y$ for $t > 0$.

- Find \vec{E} and \vec{B} everywhere for $t < 0$.
- Calculate $A_y(0, 0, z, t)$ (i.e., $x = y = 0$) from the Liénard-Wiechert retarded integral solution for the potential,

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}', t_r) d^3x'}{|\vec{x} - \vec{x}'|}.$$

(You must obtain an explicit solution and not leave your answer as an integral.) Plot A_y as a function of t for fixed z (for both $z < 0$ and $z > 0$) and as a function of z for fixed $t > 0$. Be sure to label the positions where the slope of A_y is discontinuous.

- Find \vec{E} and \vec{B} for $t > 0$ at $\vec{x} = (0, 0, z)$. You might find it helpful to express your answers using the Heaviside function, defined by $\theta(u) = 1$ for $u > 0$ and $\theta(u) = 0$ for $u < 0$.

Multiple Choice Questions (5 points each, no partial credit)

4. Two spheres of linear magnetic material are placed in a uniform external magnetic field. One sphere is paramagnetic, with $\mu > \mu_0$. The other is diamagnetic, with $\mu < \mu_0$. Which of the following is correct?

(a) The B -field inside the paramagnetic sphere is stronger than inside the diamagnetic sphere.

(b) The field is stronger inside the diamagnetic sphere.

(c) The field is the same inside the two spheres.

(d) Because of hysteresis, the answer is history-dependent.

5. You are given three differently shaped objects made of the same dielectric material with $\epsilon > \epsilon_0$ – a sphere, a needle, and a wafer (a thin disk). If an external electric field is turned on (oriented along the needle and perpendicular to the wafer), which shape has the smallest internal electric field? Choose the correct answer:

(a) The sphere.

(b) The needle.

(c) The wafer.

(d) They all have the same internal electric field.

6. A magnetic field $\vec{B} = kt\vec{e}_z$ passes through a circular disk of radius R lying in the plane $z = 0$, where $k > 0$. The disk has a net positive charge. At $t = 0$ it is at rest. At $t > 0$, which of the following is correct?

(a) Magnetic forces cause the disk to begin rotating clockwise, as viewed from above.

(b) Magnetic forces cause the disk to begin rotating counter-clockwise, as viewed from above.

(c) Electric forces cause the disk to begin rotating clockwise, as viewed from above.

(d) Electric forces cause the disk to begin rotating counter-clockwise, as viewed from above.

There are two more problems on the next page.

7. A cylindrical wire made of imperfect conductor is connected to the two terminals of a battery so that current flows through the wire. Which of the following is true?

- (a) There is a Poynting flux in the wire and its direction is parallel to the current.
- (b) There is a Poynting flux in the wire and its direction is radially outward (away from the central axis of the wire).
- (c) There is a Poynting flux in the wire and its direction is radially inward (toward the central axis of the wire).
- (d) There is no Poynting flux in the wire.

8. An electromagnetic wave with $\vec{E} = E_0 \cos(kz - \omega t)\vec{e}_x$ is incident upon a dielectric with $\epsilon \neq \epsilon_0$ and $\mu = \mu_0$. Which of the following is true?

- (a) The transmitted wave has wavenumber $k(\epsilon/\epsilon_0)$ and frequency ω .
- (b) The transmitted wave has wavenumber k and frequency $\omega(\epsilon/\epsilon_0)$.
- (c) The transmitted wave has wavenumber k and frequency $\omega\sqrt{\epsilon/\epsilon_0}$.
- (d) The transmitted wave has wavenumber $k\sqrt{\epsilon/\epsilon_0}$ and frequency ω .