Massachusetts Institute of Technology
Department of Physics

Physics 8.07 Fall 2005

Problem Set #10
Due at 4pm Fri, December 2, 2005

1. **EM Waves in a Plasma**

The Drude-Lorentz model presented in lecture gives the conductivity

$$\sigma = \frac{n_e e^2}{m(\gamma_0 - i\omega)}$$

where n_e is the number density of free electrons, m is the electron mass, γ_0 is a damping rate, and ω is the angular frequency. In this problem we assume $\omega \gg \gamma_0$ and $\omega \gg \omega_j$ for all j where ω_j are the resonant frequencies of the bound electrons. Under these conditions all the electrons behave like free electrons, i.e. the material behaves like a plasma. This description holds not only for fully ionized gases but also for electrons in a metal at optical and higher frequencies.

a) Show that the Maxwell equations in the material with conductivity σ admit transverse plane wave solutions with dispersion relation

$$\omega^2 = \omega_p^2 + k^2 c^2$$

and find the plasma frequency ω_p in terms of n_e, e, m, and ϵ_0. Show that this dispersion relation follows from the high-frequency limit of Griffiths Eq. (9.161).

b) Assume that space is vacuum for $z < 0$ and plasma for $z > 0$. A monochromatic plane wave of frequency $\omega < \omega_p$ in the vacuum travels in the z-direction. Derive the complex amplitudes of the reflected and transmitted electric fields $\tilde{\mathcal{E}}_r$ and $\tilde{\mathcal{E}}_t$ in terms of the amplitude of the incident electric field $\tilde{\mathcal{E}}_i$.

c) Derive the reflection and transmission coefficients and check that they add up to 1. (Hint: be very careful in taking the time average of the Poynting flux to obtain the intensity.) What happens to the wave in the plasma? How would this change if $\omega > \omega_p$?

2. **Split-Ring Resonators and Negative Index Materials (20 points)**
Recently artificial materials have been designed having simultaneously both $\epsilon < 0$ and $\mu < 0$ resulting in a negative index of refraction (J.B. Pendry and D.R. Smith, *Physics Today*, June 2004, pp. 37–43). Negative permittivity can be achieved for low-loss materials at frequencies just above the resonant frequency, according to Griffiths Eq. (9.161). Negative permeability is harder to achieve. Pendry and Smith used a lattice of split-ring resonators (SRR) to achieve this end. This problem computes the magnetic properties of a SRR and then investigates the peculiar optical properties of negative-index materials.

a) Consider a plane electromagnetic wave with $\vec{H} = \mu_0^{-1}\vec{B} = \mathrm{Re}\{\tilde{H}\exp[i(kz - \omega t)]\}\vec{e}_y$ incident on a small, thin circular conducting ring of radius r and resistance R centered at $\vec{x} = 0$ and oriented perpendicularly to $\vec{e}_y$. Assuming that (1) $r \ll 2\pi/k$, (2) the ring's self-inductance is negligible, and (3) the Kirchoff loop rule holds(i.e., the sum of the EMFs around a closed loop vanishes), show that the induced magnetic moment of the ring can be written $\vec{m} = \mathrm{Re}\{\tilde{m}\exp(-i\omega t)\}\vec{e}_y$ and find the relation between $\tilde{m}$ and $\tilde{H}$.

b) At high frequencies the Kirchoff loop rule breaks down because the inertia of the charges cannot be neglected (i.e., the charges do not have sufficient time to achieve terminal velocity before the electric field changes). Instead, the charges in the ring obey the equation of motion

$$\frac{d^2\vec{x}}{dt^2} + \gamma\frac{d\vec{x}}{dt} = \frac{q}{m}\vec{E}$$

where γ is a damping coefficient which gives rise to the ring's resistance. Let ϕ be the angle around the ring and let the ring contain λ free charges per unit length where λ is a constant. Assuming that all the charges have the same velocity $r\dot{\phi}$, the current in the ring is $I = \lambda q r\dot{\phi}$. Using the equation of motion of the charges, find the modified relation between $\tilde{m}$ and $\tilde{H}$. As a check on your result, you should find that if the acceleration (i.e., inertia) of the charges is neglected, comparing your result with part (a) yields $R = 2\pi r m\gamma/(\lambda q^2)$.

c) Now suppose that a small cut is made in the circular ring making it into a shape like the letter C except with a much smaller gap (this is the "split" of the split ring) so that it behaves like a ring with a capacitor. Now the electric field in the ring has two parts: (1) a transverse (solenoidal, or curl-type) field coming from $-\partial\vec{B}/\partial t$ and a longitudinal (potential, or gradient-type) field coming from the charges accumulating on either side of the cut in the ring. Show that

$$2\pi r E_\phi = -\frac{d}{dt}(\pi r^2 B_y) - \frac{Q}{C}$$

where $I = dQ/dt$ and C is the capacitance of the gap in the ring.

d) By applying the methods of part (b) to the split ring, show that

$$\tilde{m} = -\frac{A\omega^2 \tilde{H}}{\omega^2 - \omega_0^2 + i\gamma\omega}$$

and find A and ω_0^2. This result shows that the current in the split ring oscillates as in a damped, driven resonator.

e) Now suppose that a cubic lattice of the split-ring resonators is made with N SRR per unit volume and that the lattice spacing is much smaller than a wavelength so that the material appears to the electromagnetic wave to be homogeneous. Define the complex permeability by $B = \mathrm{Re}\{\tilde{\mu}\tilde{H}\exp[i(kz - \omega t)]\}$. Show that

$$\tilde{\mu} = \mu_0 \left(1 - \frac{NA\omega^2}{\omega^2 - \omega_0^2 + i\gamma\omega}\right) .$$

f) Finally, consider a material for which ϵ and μ are both real and negative. Show that the index of refraction is negative. Consider a plane wave travelling through vacuum incident upon a block of material with index n. For an angle of incidence $\theta_i = \pi/4$, determine the angle of the refracted (transmitted) ray, if it exists, for the cases $n = 1.5$, $n = 0.5$, and $n = -1$. Make a sketch of the light ray as it gets refracted in a medium with $n = -1$. What shape would one have to make a lens of negative-index material to focus light?

3. **Griffiths Problem 9.31 (p. 412)**
   c) In part (b), what is I/λ? Explain why this does not violate special relativity.

4. **Griffiths Problems 10.3 and 10.5 (p. 420)**

5. **Griffiths Problem 10.9 (p. 426)**