Massachusetts Institute of Technology
Department of Physics

Physics 8.07 Fall 2005

# Final Exam

## 1. Radiation from a classical atom (25 points)

The Bohr atom consists of a classical electron of mass $m$ orbiting a stationary proton. The fine structure constant is $\alpha = e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$.

a) The ground state corresponds to a circular orbit of angular momentum $\hbar$. Find the radius $a_0$ and speed $v$ for this orbit and show that $v/c \ll 1$.

b) If the electron's orbital angular momentum is $\hbar\vec{e}_z$ and its angular frequency is $\omega = \hbar/(ma_0^2)$, determine the electron's acceleration $\vec{a}(t)$ and $\hat{r}\times\vec{a}(t)$ where $\hat{r} = \vec{e}_x \sin\theta\cos\phi + \vec{e}_y \sin\theta\sin\phi + \vec{e}_z\cos\theta$ points from the proton to the observation point. Write your results in terms of $t$, $\theta$, $\phi$, $a_0$, and $\omega$.

c) At large distance $r = |\vec{x}\,|$ from the electron the electric field is well approximated by

$$\vec{E}(\vec{x}, t) \approx -\frac{\mu_0 e}{4\pi r}[\hat{r} \times (\hat{r} \times \vec{a}\,)] \ .$$

In terms of $c$ and the quantities appearing in this formula, write down $\vec{B}(\vec{x}, t)$ and the Poynting vector $\vec{S}(\vec{x}, t)$.

d) Using the results of parts (b) and (c), determine the time-average power per unit solid angle $dP/d\Omega$ radiated by the electron.

e) Integrate $dP/d\Omega$ to get the total power $P$ radiated by the electron. Given that the electronic energy is

$$E = \frac{e^2}{8\pi\epsilon_0 a_0} \ ,$$

a characteristic lifetime $T = E/P$ can be defined for the classical atom. Find $T$ and express it in terms of $\omega$ and $\alpha$. Given that $2\pi/\omega = 3 \times 10^{-16}$ s for the ground state of the Bohr atom, estimate the atomic lifetime $T$ to within an order of magnitude. (The ground state of the real hydrogen atom is, of course, described not by a classically orbiting particle but by a stationary wave.)

2. **Cylindrical cable (25 points)**
A cylindrical cable consists of two very long, conducting concentric circular cylinders centered on the $z$-axis. The radii of the two cylinders are $s = a$ and $s = b$ in cylindrical coordinates where $a < b$. The permittivity is $\epsilon_0$ everywhere.

a) Compute the capacitance per unit length.

b) For the rest of this problem there is no charge on the conductors. A current $I(t)$ flows down the inner conductor (i.e., along the $z$-axis) and returns back along the outer conductor. The currents are uniformly spread over the two conductors. Neglecting the displacement current, find the magnetic field everywhere.

c) In what *direction* does the induced electric field point (radial, circumferential, or along the $z$-axis)?

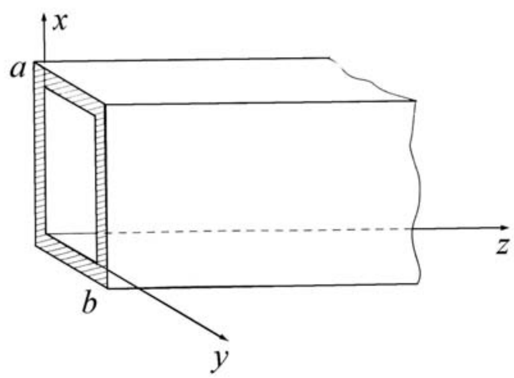
d) Assuming that the field goes to zero as $s \to \infty$, find $\vec{E}(s, t)$ everywhere. (Hint: be very careful across $s = a$ and $s = b$. Do not be alarmed if you find $\vec{E} \neq 0$ in a conductor; by approximating the cable as being very long and the current changing simultaneously throughout we implicitly neglect the rearrangement of charges to cancel $\vec{E}$ in the conductor.)

e) The displacement current density is

$$\vec{J}_{\text{disp}} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} .$$

Find the displacement current $I_{\text{disp}}(s, t)$ passing through a disk of radius $s < a$. If $I(t) = I_0 \cos \omega t$, what is the condition on $\omega$ so that $I_{\text{disp}} \ll I$?

3. **Fields in a rectangular waveguide (25 points)**




In this problem we consider TM waves ($B_z = 0$) in a wave guide of rectangular shape (see figure), with height $a$ and width $b$. Griffiths shows that all fields can be obtained from $E_z = E_{z0} e^{i(kz-\omega t)}$ and that $E_{z0}(x, y)$ can be determined as a solution to the following wave equation:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_{z0} = 0 \ . \qquad (1)$$

a) What are the boundary conditions for $E_{z0}(x, y)$?

b) Assume a solution on the form $E_{z0} = X(x)Y(y)$. Write down two ordinary differential equations to determine $X(x)$ and $Y(y)$, respectively.

The general solutions for $X(x)$ and $Y(y)$ are

$$\begin{aligned} X(x) &= A_x \sin(k_x x) + B_x \cos(k_x x) \quad , \qquad (2) \\ Y(y) &= A_y \sin(k_y y) + B_y \cos(k_y y) \quad . \end{aligned}$$

c) Write down expressions for the particular solutions of $E_{z0}$ that satisfy the boundary conditions. Also, write down the dispersion relations corresponding to these solutions.

One solution of $E_{z0}$ consistent with the boundary conditions is given by

$$E_{z0} = \sin(\pi x/a) \sin(\pi y/b) \quad . \qquad (3)$$

d) What is the lowest frequency, $\omega_{11}$, for which this solution propagates?

e) What are the group and phase velocities corresponding to the solution in Eq. 3 ?

**Multiple Choice Questions (5 points each, no partial credit)**

4. A free surface current circulates around a long cylinder so as to create a magnetic field $\vec{B}_0$ inside the cylinder. Then the cylinder is filled with a linear magnetic material with permeability $\mu$ without changing the free surface current on the cylinder. The magnetic field in the cylinder becomes

(a) $(\mu_0/\mu)\vec{B}_0$.
(b) $(\mu/\mu_0)\vec{B}_0$.
(c) $(2\mu_0/3\mu)\vec{B}_0$.
(d) $(2\mu/3\mu_0)\vec{B}_0$.

5. A parallel plate capacitor consists of two oppositely charged plates of area $A$ parallel to the $x$-$y$ plane and located at $z = \pm\frac{1}{2}d$. At distances $r \gg \max(d, \sqrt{A})$ from the capacitor, the electric field

(a) varies as $E \propto r^{-3}$ and is strongest along the $z$-axis.
(b) varies as $E \propto r^{-3}$ and is strongest in the $x$-$y$ plane.
(c) varies as $E \propto r^{-4}$ and is strongest along the $z$-axis.
(d) varies as $E \propto r^{-4}$ and is strongest in the $x$-$y$ plane.

6. Two identical dipole magnets, both aligned along the $z$-direction, are held apart with separation $s$ in the $x$-$y$ plane. The force each magnet exerts on the other varies with $s$ as

(a) $F \propto s^{-2}$ and is attractive.
(b) $F \propto s^{-3}$ and is attractive.
(c) $F \propto s^{-3}$ and is repulsive.
(d) $F \propto s^{-4}$ and is repulsive.

**There are two more problems on the next page.**

7. Which one of these statements is true?

(a) Electric fields are always perpendicular to a perfectly conducting surface, even in the time-dependent case.

(b) Field lines of $\vec{H}$ can have no ends but may form closed loops.

(c) In static situations, field lines of $\vec{J}$ can have no ends but may form closed loops.

(d) Field lines of $\vec{D}$ cannot form closed loops.

8. The complex permittivity of a material is $\tilde{\epsilon} = \epsilon_0(a + ib)$ where $a$ and $b$ are real numbers. The phase velocity is

(a) $v = ac$.

(b) $v = c/\sqrt{a}$.

(c) $v = c/\sqrt{a^2 + b^2}$.

(d) $v = \left(\frac{2c^2}{a+\sqrt{a^2+b^2}}\right)^{1/2}$.