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Electromagnetic Radiation and Scattering

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1 Lorentz Gauge Potentials in the Far-Field Region

In Lorentz gauge, the potentials $V(\vec{x},t) \equiv cA^0(\vec{x},t)$ and $A_i(\vec{x},t) \equiv A^i(\vec{x},t)$ (where $i \in \{1,2,3\}$) solving the Maxwell equations are

$$A^\mu(\vec{x},t) = \frac{\mu_0}{4\pi}\int \frac{J^\mu(\vec{x}\,',t_r)}{|\vec{x}-\vec{x}\,'|}\,d^3x' \quad \text{where} \quad t_r(t,\vec{x},\vec{x}\,') \equiv t - \frac{|\vec{x}-\vec{x}\,'|}{c} \ , \tag{1}$$

with $\mu \in \{0,1,2,3\}$, $J^0(\vec{x},t) \equiv c\rho(\vec{x},t)$ and $J^i(\vec{x},t) \equiv J_i(\vec{x},t)$. The similarity of the wave equations for the scalar and vector potentials allows us to combine their solutions in one compact formula. Later we will see that the 4-vectors A^μ and J^μ are much more than simply a convenient shorthand notation.

We will evaluate the integral in equation (1) making two approximations. The first one is that the observer is very distant compared with both the size of the source and the wavelength of light — so distant, in fact, that only the parts of $\vec{E}$ and $\vec{B}$ that decline as $1/r$ with distance are retained. This excludes the static fields and allows us to replace $|\vec{x}-\vec{x}\,'|$ by $r \equiv |\vec{x}\,|$ in the denominator of the integral. This far-field restriction also allows us to approximate the retarded time by

$$t_r \approx t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{x}\,'}{c} \equiv t_0 + \frac{\hat{r}\cdot\vec{x}\,'}{c} \ , \quad t_0 \equiv t - \frac{r}{c} \ , \tag{2}$$

where $\hat{r}$ is the unit vector in the direction of $\vec{x}$, i.e. $\vec{x} = r\hat{r}$. (The r and $\hat{r}$ notation is similar to that used by Griffiths.) The retarded time from different points in the source varies only slightly about the value t_0 at the origin.

Second, we assume that the source charges are slowly varying (equivalently, the charges move nonrelativistically) so that the Taylor expansion of $J^\mu(\vec{x},t_r)$ about $t_r = t_0$ converges rapidly. In that case we may write

$$J^\mu(\vec{x}\,',t_r) \approx J^\mu(\vec{x}\,',t_0) + \frac{\hat{r}\cdot\vec{x}\,'}{c}\dot{J}^\mu(\vec{x}\,',t_0) + \frac{1}{2}\left(\frac{\hat{r}\cdot\vec{x}\,'}{c}\right)^2 \ddot{J}^\mu(\vec{x}\,',t_0) + \cdots \ . \tag{3}$$

For a source varying sinusoidally with angular frequency ω, $\dot{J}^\mu \sim \omega J^\mu$ (to order of magnitude) so the Taylor series converges rapidly if $\omega r'/c \ll 1$ which is satisfied if the source varies slowly compared with the time for light to cross it. This condition also implies $r' \ll \lambda = 2\pi c/\omega$. Thus our two assumptions combined may be written

$$r' \ll \lambda \ll r \ . \tag{4}$$

The potentials are now given by integrating equation (3) over volume. Integrating $J^0 = c\rho$ gives

$$4\pi\epsilon_0 r V(\vec{x}, t) \approx Q + \frac{\hat{r}}{c} \cdot \frac{d\vec{p}}{dt}(t_0) + \frac{\hat{r}_i \hat{r}_j}{2c^2} \frac{d^2 C_{ij}}{dt^2}(t_0) + \cdots \ , \tag{5}$$

where

$$Q \equiv \int \rho' \, d^3x' \ , \quad \vec{p}(t_0) \equiv \int \vec{x}\,'\rho' \, d^3x' \ , \quad C_{ij}(t_0) \equiv \int x'_i x'_j \rho' \, d^3x' \tag{6}$$

are the first three moments of the charge distribution[1] and $\rho' \equiv \rho(\vec{x}\,', t_0)$. Note that charge conservation implies Q is constant. The static potential $Q/(4\pi\epsilon_0 r)$ gives a $1/r^2$ static field instead of radiation, so we drop this term from further consideration.

The moments of the current density require more work. They may be evaluated using the methods introduced in Lecture 18 for magnetostatics. The zeroth moment (volume integral of $\vec{J}\,'$) is found as follows:

$$\begin{aligned} \int x'_i \left(\frac{\partial J'_j}{\partial x'_j} \right) d^3x' &= -\int x'_i \dot{\rho}' \, d^3x' = -\frac{dp_i}{dt} \\ = \int \left[\frac{\partial}{\partial x'_j}(x'_i J'_j) - J'_i \right] d^3x' &= -\int J'_i \, d^3x' \ , \end{aligned}$$

where in the first line we used charge conservation $\dot{\rho} + \partial J_j/\partial x_j = 0$ (recall the Einstein summation convention) and in the second line we used the fact that the source is localized while the integral is over all space. From this we conclude

$$\int J'_i \, d^3x' = \frac{dp_i}{dt}(t_0) \ . \tag{7}$$

Similarly,

$$\int x'_i x'_j \left(\frac{\partial J'_k}{\partial x'_k} \right) d^3x' = -\frac{dC_{ij}}{dt} = \int \left[\frac{\partial}{\partial x'_k}(x'_i x'_j J'_k) - (x'_i J'_j + x'_j J'_i) \right] d^3x' \ . \tag{8}$$

For convenience we define

$$D_{ij} \equiv \int x'_i J'_j \, d^3x' \ .$$

[1] Moments of a distribution are integrals over the distribution multiplied by one or more factors of the integration variable.

Then the antisymmetric part of D_{ij} follows from

$$\begin{aligned} D_{ij} - D_{ji} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \int x'_l J'_m \, d^3x' = \epsilon_{ijk}\epsilon_{lmk} \int x'_l J'_m \, d^3x' \\ = \quad \epsilon_{ijk} \int (\vec{x}\,' \times \vec{J}\,')_k \, d^3x' = 2\epsilon_{ijk} m_k \ , \end{aligned}$$

where we introduced the magnetic dipole moment,

$$\vec{m} = \frac{1}{2} \int \vec{x}\,' \times \vec{J}\,' \, d^3x' \ . \tag{9}$$

The symmetric part of D_{ij} follows from equation (8):

$$D_{ij} + D_{ji} = \frac{dC_{ij}}{dt} \ .$$

Combining the symmetric and antisymmetric parts, we find

$$\int x'_j J'_i \, d^3x' = -\epsilon_{ijk} m_k(t_0) + \frac{1}{2}\frac{dC_{ij}}{dt}(t_0) \ . \tag{10}$$

Substituting equations (3), (7), and (10) into the spatial part ($\mu \in \{1, 2, 3\}$) of equation (1) gives the vector potential:

$$\frac{4\pi r}{\mu_0} \vec{A}(\vec{x}, t) \approx \frac{d\vec{p}}{dt}(t_0) - \frac{\hat{r}}{c} \times \frac{d\vec{m}}{dt}(t_0) + \frac{1}{2c}\frac{d^2\vec{C}}{dt^2}(t_0) + \cdots \ , \tag{11}$$

where $C_i \equiv \hat{r}_j C_{ij}$. Although the electric and magnetic dipole terms appear similar on the right-hand side, the electric dipole term came from the zeroth-order term in the Taylor series expansion of the current while the magnetic dipole and C_{ij} terms both came from the first-order term. For nonrelativistic sources we therefore expect magnetic dipole and electric quadrupole radiation to be weaker than electric dipole radiation.

2 Electric and Magnetic Fields for Radiation from Non-Relativistic Extended Sources

Having worked out the potentials by integrating the charge and current distributions over volume, we are prepared to evaluate the electric and magnetic fields using

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \ , \quad \vec{B} = \vec{\nabla} \times \vec{A} \ . \tag{12}$$

The potentials depend on $\vec{x}$ in two ways: (1) through the $1/r$ and $\hat{r}$ factors, and (2) through the retarded time $t_0 = t - r/c$. The first dependence leads to fields that fall off as $1/r^2$ or faster. The gradient of the retarded time, however, preserves the $1/r$

dependence because $\vec{\nabla} t_0 = -\hat{r}/c$. Likewise, all of the time-dependence of the potentials comes from the retarded time.

Radiation fields exist because of retarded time! To get the $1/r$ fields we need consider only the spacetime dependence of retarded time, $t_0(t, r) = t - r/c$. The scalar and vector potentials have several terms contributing to radiation, which we consider separately below.

2.1 Electric Dipole Radiation

Electric dipole radiation arises from the terms in equations (5) and (11) that are proportional to $d\vec{p}/dt$. The corresponding electric and magnetic fields are

$$\vec{E}_{1e} = \frac{\mu_0}{4\pi r} \hat{r} \times \left(\hat{r} \times \frac{d^2\vec{p}}{dt^2} \right) \ , \quad \vec{B}_{1e} = \frac{\hat{r}}{c} \times \vec{E}_{1e} = -\frac{\mu_0}{4\pi r} \frac{\hat{r}}{c} \times \frac{d^2\vec{p}}{dt^2} \ . \tag{13}$$

The subscript $1e$ denotes that first-order (in the Taylor series expansion of eq. 3) electric multipole. The angular distribution of radiated power is

$$\frac{dP_{1e}}{d\Omega} \equiv r^2 S_r = \frac{\hat{r} \cdot (\vec{E}_{1e} \times \vec{B}_{1e})}{\mu_0} = \frac{\mu_0}{16\pi^2 c} \left| \hat{r} \times \frac{d^2\vec{p}}{dt^2} \right|^2 = \frac{\mu_0 \ddot{p}^2}{16\pi^2 c} \sin^2\theta \ . \tag{14}$$

Here, $\ddot{p} \equiv |d^2\vec{p}/dt^2|$, θ is the angle between $\vec{p}$ and $\hat{r}$, and all quantities are to be evaluated at the retarded time t_0. The total power emitted by dipole radiation gives the *Larmor formula*,

$$P_{1e} \equiv \int \frac{dP_{1e}}{d\Omega} \, d\Omega = \frac{\mu_0 \ddot{p}^2}{6\pi c} \ . \tag{15}$$

For a harmonically varying dipole, i.e. $p(t) = p_0 \cos\omega t$, the time average is $\langle \ddot{p}^2 \rangle = \frac{1}{2} p_0^2 \omega^4$. Most transitions between bound states of atoms can be described as emission or absorption of electric dipole radiation resulting from a time-varying dipole moment of the charge distribution. The correct treatment of this process requires quantum mechanics because the amount of radiated energy is not arbitrary but is quantized into discrete amounts corresponding to the differences in the atomic energy levels. However, the classical theory is remarkably accurate once allowance is made for the discrete resonant frequencies.

2.2 Magnetic Dipole Radiation

Magnetic dipole radiation arises from the terms in equations (5) and (11) that are proportional to $d\vec{m}/dt$. The corresponding electric and magnetic fields are

$$\vec{E}_{1m} = \frac{\mu_0}{4\pi r} \frac{\hat{r}}{c} \times \frac{d^2\vec{m}}{dt^2} \ , \quad \vec{B}_{1m} = \frac{\hat{r}}{c} \times \vec{E}_{1m} \ . \tag{16}$$

This result also follows from electric dipole fields under the duality transformation mentioned in Griffiths, $\vec{E} \to c\vec{B}$, $\vec{B} \to -\frac{1}{c}\vec{E}$, $\vec{p} \to \frac{1}{c}\vec{m}$. This fact allows us to quickly deduce the angular distribution of radiated power and the total emitted power,

$$\frac{dP_{2m}}{d\Omega} = \frac{\mu_0}{16\pi^2 c^3} \left| \hat{r} \times \frac{d^2\vec{m}}{dt^2} \right|^2 = \frac{\mu_0 \ddot{m}^2}{16\pi^2 c^3} \sin^2\theta \ , \quad P_{2m} = \frac{\mu_0 \ddot{m}^2}{6\pi c^3} \ . \tag{17}$$

Despite its similarity to electric dipole radiation, magnetic dipole radiation is ordinarily much weaker (it came from higher order in the Taylor series expansion) because the magnetic moment comes from current rather than charge — $\vec{J} \sim \rho\vec{v}$ where $\vec{v}$ is the velocity of the charge carriers — and $m/c \sim pv/c$. Magnetic dipole radiation can be important when the electric dipole moment vanishes, as it does for certain transitions between atomic energy levels, and for strongly magnetized neutron stars (pulsars).

2.3 Electric Quadrupole Radiation

Electric quadrupole radiation arises from the terms in equations (5) and (11) that are proportional to d^2C_{ij}/dt^2. Some algebra is required to obtain simple expressions for the fields. One finds that the fields depend on C_{ij} only through $\hat{r} \times \vec{C}$ where $C_i \equiv \hat{r}_j C_{ij}$. This result is conventionally written in terms of the traceless quadrupole tensor and its projection along $\hat{r}$,

$$Q_{ij}(t_0) \equiv \int \left[3x'_i x'_j - (r')^2 \delta_{ij} \right] \, d^3x' = 3C_{ij} - C_{kk}\delta_{ij} \ , \quad Q_i \equiv \hat{r}_j C_{ij} \ . \tag{18}$$

With this definition, $\hat{r} \times \vec{C} = \frac{1}{3}\hat{r} \times \vec{Q}$. The result for the fields is

$$\vec{E}_{2e} = \frac{\mu_0}{24\pi r c}\hat{r} \times \left(\hat{r} \times \frac{d^3\vec{Q}}{dt^3} \right) \ , \quad \vec{B}_{2e} = \frac{\hat{r}}{c} \times \vec{E}_{2e} = -\frac{\mu_0}{24\pi r c^2}\hat{r} \times \frac{d^3\vec{Q}}{dt^3} \ . \tag{19}$$

The resulting angular distribution of emitted power is

$$\frac{dP_{2e}}{d\Omega} = \frac{\mu_0}{576\pi^2 c^3} \left| \hat{r} \times \frac{d^3\vec{Q}}{dt^3} \right|^2 \ . \tag{20}$$

The dependence on angles is hidden inside

$$\hat{r} = \vec{e}_x \sin\theta\cos\phi + \vec{e}_y \sin\theta\sin\phi + \vec{e}_z \cos\theta \ . \tag{21}$$

To make the angular dependence of quadrupole radiation explicit, we need to know the form of Q_{ij}. Quadrupoles can be made by superposing dipoles of equal strength but opposite orientation. The general case is a linear combination of two configurations, collinear and X.

In the collinear configuration the two dipoles lie along the same axis (Griffiths Problem 11.11). Taking the dipoles to lie along the z-axis, one finds that all elements of C_{ij} vanish except $C_{zz} \equiv \frac{1}{3}Q_{\rm coll}(t_0)$, which gives

$$[Q_{ij}] = \frac{1}{3}Q_{\rm coll}(t_0)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \Rightarrow \quad \hat{r}\times\vec{Q}_{\rm coll} = -\frac{3}{2}Q_{\rm coll}(t_0)\sin 2\theta\,\vec{e}_\phi \ . \tag{22}$$

In the X configuration the two dipoles are placed side by side (pointing in opposite directions). Taking the two dipoles to lie in the x-y plane, the nonzero elements of C_{ij} are $C_{xy} = C_{yx} \equiv \frac{1}{3}Q_{\rm X}(t_0)$, which gives

$$[Q_{ij}] = Q_{\rm X}(t_0)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \hat{r}\times\vec{Q}_{\rm X} = Q_{\rm X}(t_0)\sin\theta(-\vec{e}_\theta\cos 2\phi + \vec{e}_\phi\cos\theta\sin 2\phi) \ . \tag{23}$$

It should be noted that the general quadrupole tensor has 5 linearly independent coefficients (a symmetric 3×3 matrix has six but the trace is zero). There are two linearly independent collinear configurations described by three diagonal elements whose sum vanishes; an independent collinear configuration may be obtained simply by rotating the charges to lie along a different axis than the z-axis. There are three linearly independent X configurations, namely those in which the dipoles lie in the three orthogonal planes x-y, y-z, and x-z. Therefore, aside from rotation of the coordinates, the two cases considered are fully general.

The angular power pattern for the quadrupole has more nodes (directions along which the radiated power vanishes) than for dipole radiation. For example, for the collinear pattern, $dP/d\Omega \propto \sin^2(2\theta)$ vanishes for $\theta = 0$ and $\theta = \pi/2$ while radiation is maximal along the cones $\theta = \pi/4$ and $\theta = 3\pi/4$. The X pattern is not axisymmetric; the radiation is peaked into four lobes equally spaced in azimuth.

To get the total power we integrate equation (20) over solid angle. This is tricky because of the angular dependences associated with all the $\hat{r}$ unit vectors. The integration is most easily done by writing the square of the vector appearing in the formula using index notation,

$$\begin{aligned} \left|\hat{r}\times\frac{d^3\vec{Q}}{dt^3}\right|^2 &= \left(\epsilon_{ijk}\hat{r}_j\hat{r}_l\frac{d^3Q_{kl}}{dt^3}\right)\left(\epsilon_{imn}\hat{r}_m\hat{r}_o\frac{d^3Q_{no}}{dt^3}\right) \\ &= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\hat{r}_j\hat{r}_l\hat{r}_m\hat{r}_o\frac{d^3Q_{kl}}{dt^3}\frac{d^3Q_{no}}{dt^3} \\ &= \hat{r}_l\hat{r}_m\frac{d^3Q_{kl}}{dt^3}\frac{d^3Q_{km}}{dt^3} - \hat{r}_j\hat{r}_k\hat{r}_l\hat{r}_m\frac{d^3Q_{jk}}{dt^3}\frac{d^3Q_{lm}}{dt^3} \ . \end{aligned}$$

Now we use the angular averages

$$\langle\hat{r}_l\hat{r}_m\rangle = \frac{1}{3}\delta_{lm} \ , \quad \langle\hat{r}_j\hat{r}_k\hat{r}_l\hat{r}_m\rangle = \frac{1}{15}(\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}) \tag{24}$$

to conclude

$$\frac{1}{4\pi}\int\left|\hat{r}\times\frac{d^3\vec{Q}}{dt^3}\right|^2 d\Omega=\left(\frac{1}{3}-\frac{2}{15}\right)\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}=\frac{1}{5}\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}$$

where the Einstein summation convention is used (and we used $Q_{kk}=0$). Thus the total power emitted by quadrupole radiation is

$$P_{2e}=\frac{\mu_0}{720\pi c^3}\left|\frac{d^3Q_{ij}}{dt^3}\right|^2\ , \tag{25}$$

where the square of d^3Q_{ij}/dt^3 is understood also to involve summing over the pairs of indices.

The derivation of quadrupole radiation is complicated. To go to higher order in the Taylor expansion of equation (3) it is better to represent the potentials using an expansion in spherical waves composed of spherical harmonics and radial basis functions. This technique is used in more advanced treatments. Dipole radiation corresponds to spherical harmonics $Y_{lm}(\theta,\phi)$ with $l=1$ and $m\in\{-1,0,1\}$ — the three orientations of a dipole are related to the three different m-values. Similarly, quadrupole radiation with its five degrees of freedom correspond to the 5 m-values for Y_{2m}.

Equation (25) is rarely used because electric dipole radiation is almost always stronger than quadrupole radiation. Its greatest interest lies in its similarity to the power radiated by gravitational radiation of slowly-moving sources in general relativity. In electromagnetism there is no monopole radiation because of charge conservation (recall Q is independent of retarded time in eq. 5). In general relativity there is no monopole radiation because of conservation of energy, and there is no dipole radiation because of conservation of momentum.[2] The lowest-order of gravitational radiation is given by a gravitational quadrupole radiation formula strikingly similar to equation (25):

$$P_{2G}=\frac{G}{45c^5}\left|\frac{d^3Q_{ij}}{dt^3}\right|^2\ , \tag{26}$$

where G is Newton's constant and Q_{ij} is defined using the mass density instead of the charge density.

3 Scattering of Electromagnetic Radiation

Electromagnetic radiation incident upon charges causes them to oscillate. Previously we considered the oscillations of a system of harmonically bound electrons in order to work out the frequency dependence of permittivity. If the radiation is incident on a

[2]Maxwell's equations enforce charge conservation. Einstein's equations enforce energy-momentum conservation.

single charge or atom the oscillation of the localized charges will generate electromagnetic radiation. This process is called scattering. We use the Lorentz model to derive Rayleigh scattering (scattering by bound charges) and Thomson scattering (scattering of free charges). We also examine the polarization of light caused by scattering.

Consider a plane wave incident upon a charge q initially located at $\vec{x} = 0$. The electric field of the incident wave is

$$\vec{E}_{\rm in} = \text{Re}\{\tilde{\mathcal{E}}_{\rm in} e^{i(kz-\omega t)}\vec{\epsilon}\} \ . \tag{27}$$

We use the symbol $\vec{\epsilon}$ to denote a general complex polarization vector normalized so that $\vec{\epsilon}^* \cdot \vec{\epsilon} = 1$. For light linearly polarized along the x-axis, $\vec{\epsilon} = \vec{e}_x$. Circularly polarized light has $\vec{\epsilon} = (\vec{e}_x \pm i\vec{e}_y)/\sqrt{2}$.

The charge has equation of motion $\ddot{x}+\gamma\dot{x}+\omega_0^2 x = qE_x/m$ yielding $x(t) = \text{Re}\{\tilde{x}_0 e^{-i\omega t}\}$ with

$$(-\omega^2 - i\gamma\omega + \omega_0^2)\tilde{x}_0 = \frac{q}{m}\mathcal{E}_{\rm in} \ .$$

The oscillations of the charge imply a time-varying electric dipole moment $p_x(t) = qx(t)$ or

$$\ddot{p} = \text{Re}\left\{\frac{\omega^2 q^2/m}{\omega^2 - \omega_0^2 + i\gamma\omega}\mathcal{E}_{\rm in} e^{-i\omega t}\right\} \ . \tag{28}$$

We will compute the time average of the power radiated by this charge, which requires

$$\langle \ddot{p}^2 \rangle = \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \gamma^2\omega^2}\left(\frac{q^2}{m}\right)^2 \frac{|\mathcal{E}_{\rm in}|^2}{2} \ . \tag{29}$$

The incident intensity is

$$I_{\rm in} = \frac{|\mathcal{E}_{\rm in}|^2}{2\mu_0 c} \ . \tag{30}$$

From equation (14), the time-average radiated power per solid angle for an electron with $q = -e$ is[3]

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \left(\frac{\mu_0 e^2}{4\pi m}\right)^2 \left[\frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \gamma^2\omega^2}\right] |\vec{\epsilon}\times\hat{r}|^2 I_{\rm in} \ , \tag{31}$$

which may be written

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{d\sigma}{d\Omega} I_{\rm in} \ , \tag{32}$$

where the function $d\sigma/d\Omega$, called the *differential scattering cross section*, is

$$\frac{d\sigma}{d\Omega} = r_0^2 \left[\frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \gamma^2\omega^2}\right] |\vec{\epsilon}\times\hat{r}|^2 \ , \quad r_0 \equiv \frac{\mu_0 e^2}{4\pi m} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mc^2} \ . \tag{33}$$

[3]In the case of a complex polarization vector (e.g. circularly polarized light), $|\vec{\epsilon}\times\hat{r}|^2 \equiv (\vec{\epsilon}\times\hat{r})\cdot(\vec{\epsilon}^*\times\hat{r})$.

The length scale $r_0 = 2.818 \times 10^{-15}$ m is the *classical electron radius.* In the classical Lorentz model of an electron, the electron behaves like a spherical shell of charge of this radius.

In general the incident radiation travelling in the z-direction may consist of electric field components along both $\vec{e}_x$ and $\vec{e}_y$. Consider light scattered into the x-direction, i.e. examine outgoing radiation with $\hat{r} = \vec{e}_x$ representing a scattering angle $\theta = \pi/2$. If the incident light is polarized in the x-direction, $\vec{\epsilon} = \vec{e}_x$ and $|\vec{\epsilon} \times \hat{r}|^2 = 0$. If on the other hand the incident light is polarized in the y-direction, $\vec{\epsilon} = \vec{e}_y$ and $|\vec{\epsilon} \times \hat{r}|^2 = 1$. *Scattering causes light to be polarized and the highest degree of polarization is for right-angle scattering.*

Most light sources have very little polarization. We can describe unpolarized light as a superposition of equal intensities of uncorrelated light polarized in the x- and y-directions. Then the cross product term in equation (33) is replaced by

$$|\vec{\epsilon} \times \hat{r}|^2 \to \frac{1}{2}\left(|\vec{e}_x \times \hat{r}|^2 + |\vec{e}_y \times \hat{r}|^2\right) = \frac{1}{2}(1 + \cos^2\theta) \ ,$$

where we used equation (21). This gives the differential cross section for unpolarized scattering,

$$\frac{d\sigma}{d\Omega} = r_0^2 \left[\frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}\right] \frac{(1 + \cos^2\theta)}{2} \ . \tag{34}$$

The total cross section for scattering of unpolarized radiation follows by integrating over all angles, yielding

$$\sigma = \sigma_T \left[\frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}\right] \ , \quad \sigma_T \equiv \frac{8\pi}{3} r_0^2 \ . \tag{35}$$

For free electrons with (with $\omega_0 = \gamma = 0$) or bound electrons at frequencies far above resonance, the scattering is called *Thomson scattering* and the cross section $\sigma_T = 6.65 \times 10^{-29}$ m^2 is called the *Thomson cross section.* It gives the effective area that an electron presents to a beam of light. Note that Thomson scattering is independent of frequency — light of all wavelengths is scattered equally well by free electrons. (This breaks down for high frequencies so that $\hbar\omega$ approaches or exceeds mc^2 and relativistic quantum mechanics is necessary for a correct description of scattering.)

The scattering cross section is largest at resonance. The classical scattering process at resonance is equivalent to absorption followed by emission in quantum mechanics. The frequency dependence of the cross section near resonance matches the spectral line shape for resonance scattering and for spontaneous emission (which can be interpreted as the scattering of a virtual photon into a real one in quantum field theory).

At frequencies below resonances, $\omega \ll \omega_0$, the scattering cross section is reduced by a factor ω^4/ω_0^4 compared with the Thomson cross section. Scattering by bound charges below resonance is called *Rayleigh scattering.* The strong frequency dependence means that short-wavelength radiation is scattered more effectively than long-wavelength radiation. Scattering by oxygen and nitrogen molecules in the atmosphere makes the sky

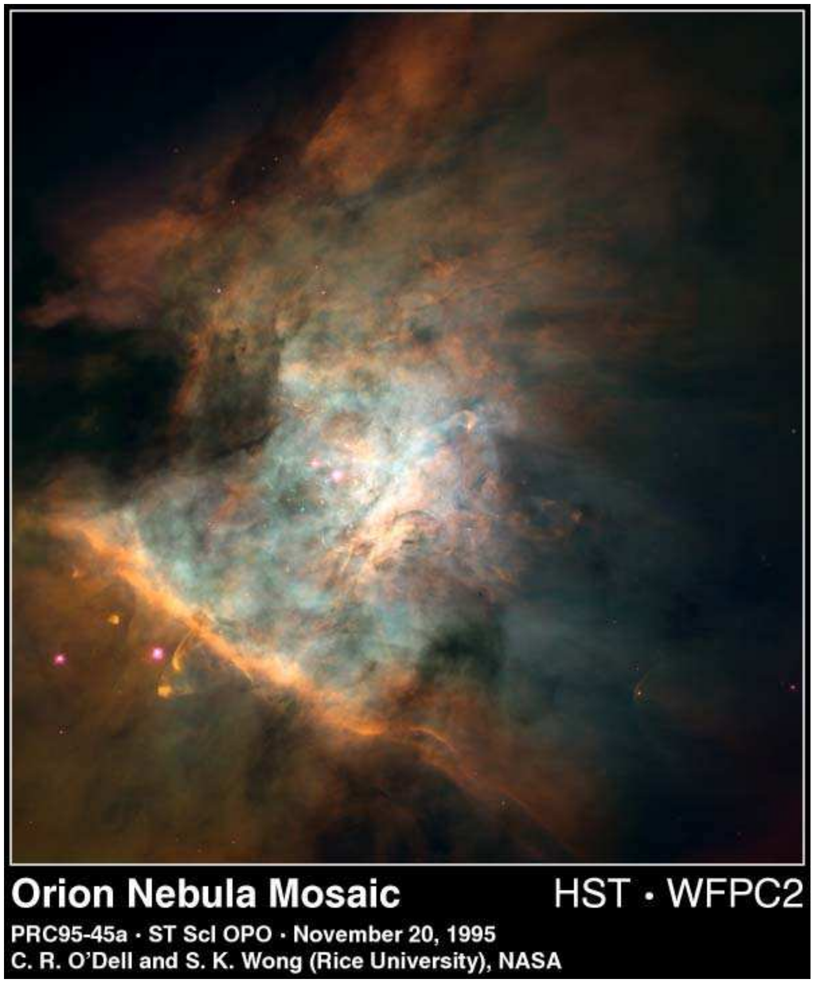

Figure 1: The center of the Orion Nebula as photographed by the Hubble Space Telescope. Image Credit: NASA/STScI, C.R. O'Dell and S.K. Wong (Rice University).

blue. Similarly, scattering in water makes swimming pools blue. A sunset is red because blue light has been scattered out of the beam. Scattering by interstellar dust grains (which, as sub-micron-sized polarizable bodies, scatter radiation somewhat like atoms or molecules do) causes interstellar clouds (nebulae, in astronomer's jargon, e.g. the Orion nebula, shown above) to appear blue. It also causes the light of background stars to be reddened.