MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — February 28, 1994
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QUIZ 1 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

a) Wright's book described the disk structure of the Milky Way.

b) Kant proposed that the faint nebulae seen in the sky are distant galaxies, similar to the Milky Way.

c) The Milky Way galaxy has a diameter of about 80,000 light-years, and a thickness of 6,000 light-years.

d) The mathematical theory of an expanding universe, in the context of general relativity, was invented by Alexandre Friedmann in 1922. (Actually the 1922 paper discussed only closed universes, but Friedmann published a second paper on open universes in 1924.) Willem de Sitter published his model of the universe in 1917. De Sitter's model was initially believed to be static, but it was later discovered that it appeared static only because it was written in peculiar coordinates— in fact it was also an expanding model. While Friedmann's equations described the general case of a homogeneous isotropic expanding universe, de Sitter's model was more specific: it was a model devoid of matter, with the expansion driven by a positive cosmological constant. The intended answer for this question was Friedmann, but full credit was given for either Friedmann or de Sitter.

e) It was Bernard Burke who told Arno Penzias about the prediction of radio noise from the big bang.

PROBLEM 2: THRESHOLD FOR PARTICLE PRODUCTION

As you were told to expect, this problem comes verbatim from the Quiz Review Problems. The solution given in the review problems is as follows:

The key relationship needed for this problem is the relativistic relation between energy, momentum, and rest mass. Using $p^0 = E/c$ and $p^2 = -m^2c^2$, one has

$$E^2 - |\vec{p}\,|^2 c^2 = M^2 c^4 \ .$$

Thus the total energy and momentum of the initial state are given by

$$E_{\text{total}} = E + m_p c^2$$

$$|\vec{p}_{\text{total}}| = \sqrt{\frac{E^2}{c^2} - m_e^2 c^2} \ .$$

The final state must of course have the same total energy and momentum, and will behave as if it were a particle of mass $m_e + 3m_p$. Thus,

$$E_{\rm total}^2 - |\vec{p}_{\rm total}|^2 c^2 = (m_e + 3m_p)^2 c^4 \ .$$

So

$$E^2 + 2Em_pc^2 + m_p^2c^4 - (E^2 - m_e^2c^4) = (m_e^2 + 6m_em_p + 9m_p^2)c^4 \ .$$

And finally,

$$\boxed{E = (4m_p + 3m_e)c^2 \ .}$$

PROBLEM 3: TRAINS, LIGHT PULSES, AND ARROWS

a) This part is done most easily in the frame of the train. In this frame the center clock and front clock are synchronized, and the pulse leaves the center clock at $t' = t_0$. The distance traveled by the light pulse is ℓ, so the reading on the front clock is

$$\boxed{t'_A = t_0 + \frac{\ell}{c} \ .}$$

ALTERNATIVE IN FRAME OF THE GROUND: For those who choose to work in the frame of the ground, the problem is more difficult. The first step is to find the x_A and t_A coordinates of the event A, as is done in part (b). Then to find the time in the (primed) frame of the train, one can use the Lorentz transformation

$$t' = t_0 + \gamma_t \left(t - \frac{v_t x}{c^2} \right) \ .$$

The term t_0 is not part of the standard form of the Lorentz transformation, but is included here because the primed clocks are initialized to read $t' = t_0$ when $x = t = 0$, instead of the customary value $t' = 0$. Inserting the expressions in part (b) for x_A and t_A into the expression above, one finds after some algebra that the boxed answer is obtained.

b) To find the trajectory of the front clock, recall that at $t = 0$ the center of the train is at $x = 0$, and the train appears to ground-observers to be compressed by the factor γ_t, where

$$\gamma_t = \frac{1}{\sqrt{1 - \frac{v_t^2}{c^2}}} \ .$$

The front clock moves with the train at speed v_t, so the trajectory is given by

$$x_{\text{front}} = \frac{\ell}{\gamma_t} + v_t t \ .$$

The trajectory of the light pulse is given by

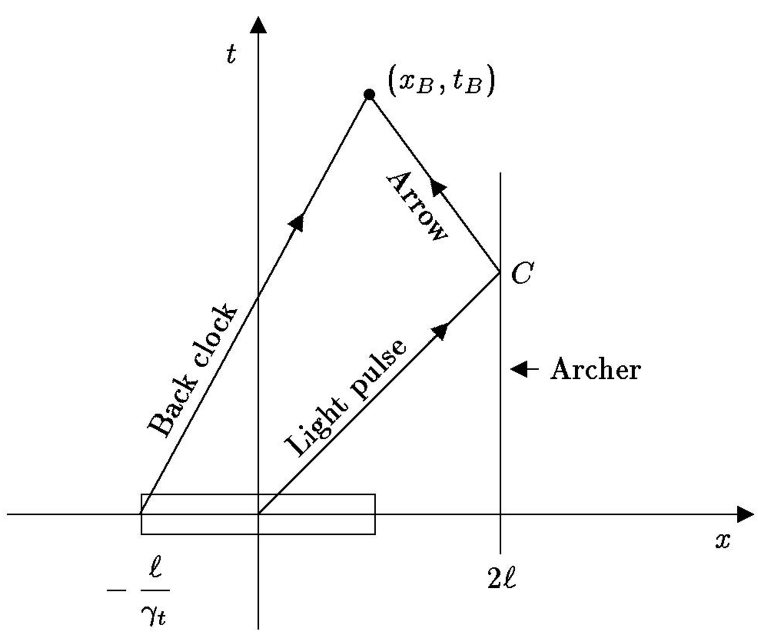
$$x_{\text{pulse}} = ct \ .$$

Solving these two trajectory equations simultaneously, the intersection occurs at

$$\boxed{\begin{aligned} t_A &= \frac{\ell}{c}\sqrt{\frac{1+\beta_t}{1-\beta_t}} \\ x_A &= \ell\sqrt{\frac{1+\beta_t}{1-\beta_t}} \ , \end{aligned}}$$

where $\beta_t = v_t/c$.

c) This part is done most simply in the frame of reference of the ground, for which the spacetime diagram looks as follows:



We have introduced the label C to denote the event when the light pulse reaches the archer. The trajectories of the light pulse and archer are given by

$$x_{\text{pulse}} = ct$$
$$x_{\text{archer}} = 2\ell \ .$$

The intersection of these two lines gives the coordinates of event C:

$$x_C = 2\ell$$
$$t_C = \frac{2\ell}{c} \ .$$

The trajectory of the arrow starts at the event C and then moves backward at speed v_a, so

$$\begin{aligned} x_{\text{arrow}} &= x_C - v_a(t - t_C) \\ &= 2\ell - v_a\left(t - \frac{2\ell}{c}\right) \ . \end{aligned}$$

To find the trajectory of the back clock, recall that at $t = 0$ the center of the train is at $x = 0$, and the train appears to ground-observers to be compressed by the factor γ_t. So

$$x_{\text{back}} = -\frac{\ell}{\gamma_t} + v_t t \ .$$

The event B is the intersection of the arrow and the back clock, so one solves the equations for these two trajectories simultaneously:

$$2\ell - v_a\left(t_B - \frac{2\ell}{c}\right) = -\frac{\ell}{\gamma_t} + v_t t_B$$

$$t_B = \frac{1}{v_a + v_t}\left[2\ell + 2\ell\frac{v_a}{c} + \frac{\ell}{\gamma_t}\right]$$

$$\boxed{t_B = \frac{\ell}{v_a + v_t}\left[2(1 + \beta_a) + \frac{1}{\gamma_t}\right] \ .}$$

where $\beta_a \equiv v_a/c$. Then

$$\begin{aligned} x_B &= -\frac{\ell}{\gamma_t} + v_t t_B \\ &= -\frac{\ell}{\gamma_t} + \frac{\ell v_t}{v_a + v_t}\left[2(1 + \beta_a) + \frac{1}{\gamma_t}\right] \end{aligned}$$

$$x_B = \frac{\ell}{v_a + v_t}\left[2(1+\beta_a)v_t - \frac{v_a}{\gamma_t}\right] .$$

d) This part is answered by using the Lorentz transformation to express t'_B in terms of x_B and t_B, which were found in the previous part. One must take into account the minor twist that in this case the primed clocks were set to $t' = t_0$ at $x = t = 0$, rather than being set to $t' = 0$. The only affect is to add t_0 to all readings on primed clocks.

$$x' = \gamma_t\,(x - v_t t)$$

$$t' = t_0 + \gamma_t\left(t - \frac{v_t x}{c^2}\right) .$$

One must as always be careful with the sign appearing in the Lorentz transformation: the sign appearing in the x' equation can be read from the diagram, since the origin $x = 0$ is moving in the negative x' direction. The sign in the t' equation is always the same as the sign in the x' equation. Formally inserting the answer from part (d):

$$t'_B = t_0 + \gamma_t\left(t_B - \frac{v_t x_B}{c^2}\right) .$$

EXTENSION: For those who want to know the final answer, algebraic simplification of the expression above gives

$$t'_B = t_0 + \frac{\ell}{v_a + v_t}\left[1 + \beta_a\beta_t + \frac{2(1+\beta_a)}{\gamma_t}\right] .$$

PROBLEM 4: TRANSFORMATION OF TRANSVERSE VELOCITIES

a) In the Star Base 12 reference system, the clocks on the Klingon warship are moving. So, from the point of view of the Star Base, the Klingon clock clicks crawlingly by a factor

$$\gamma_K = \frac{1}{\sqrt{1 - v_K^2/c^2}} .$$

So, while the Kingon clock clicks Δt_1, the Star Base clocks click $\gamma_K\,\Delta t_1$. Since the Klingon crate is cruising at speed v_K, it will cruise a distance $v_K\gamma_K\,\Delta t_1$ while the Star Base clocks click $\gamma_K\,\Delta t_1$. So

$$x_{esc} = v_K\gamma_K\,\Delta t_1$$

$$y_{esc} = 0$$

$$t_{esc} = \gamma_K\,\Delta t_1 .$$

b) In general, as it says on the front of the quiz, the four-velocity is related to the ordinary velocity by $u^i = \gamma v^i$, and $u^0 = \gamma c$. In the frame of the Klingon warship, the escape vehicle is moving in the y'-direction with speed v_{esc}. So

$$\boxed{\begin{aligned} u'^0 &= \gamma_{esc} c \\ u'^1 &= 0 \\ u'^2 &= \gamma_{esc} v_{esc} \\ u'^3 &= 0 \ , \end{aligned}}$$

where

$$\gamma_{esc} = \frac{1}{\sqrt{1 - v_{esc}^2/c^2}} \ .$$

(Since the problem involves only 2 space dimensions, no credit will be lost if you did not mention that $u'^3 = 0$.)

c) The Lorentz transformation is given by

$$\begin{aligned} u^0 &= \gamma_K (u'^0 + \beta_K u'^1) \\ u^1 &= \gamma_K (u'^1 + \beta_K u'^0) \\ u^2 &= u'^2 \\ u^3 &= u'^3 \ , \end{aligned}$$

where $\beta_K \equiv v_K/c$. Using the values of u'^μ from part (b),

$$\boxed{\begin{aligned} u^0 &= \gamma_K \gamma_{esc} c \\ u^1 &= \gamma_K \gamma_{esc} v_K \\ u^2 &= \gamma_{esc} v_{esc} \\ u^3 &= 0 \ . \end{aligned}}$$

To check yourself for mistakes, it is a good idea to verify that $u^2 \equiv (u^1)^2 + (u^2)^2 + (u^3)^2 - (u^0)^2 = -c^2$.

d) Using the Klingon coordinate system, the escape vehicle is launched at $t' = \Delta t_1$. The telemetry signal is emitted after the clocks on the escape vehicle measure an interval Δt_2, but these clocks appear to be running slowly by a factor of γ_{esc}. Thus, the time when the signal is emitted is

$$\boxed{t'_s = \Delta t_1 + \gamma_{esc}\,\Delta t_2 \ .}$$

Since the escape vehicle is moving at speed v_{esc} (in the Klingon reference system) for a time interval $\gamma_{esc}\Delta t_2$, the y' coordinate is

$$\boxed{y'_s = \gamma_{esc} v_{esc}\,\Delta t_2 \ .}$$

Since the escape vessel moves along the y' axis, it follows that

$$\boxed{x'_s = 0 \ .}$$

e) The Lorentz transformation equations can be written

$$\begin{aligned} t &= \gamma_K \left(t' + \frac{v_K x'}{c^2} \right) \\ x &= \gamma_K \left(x' + v_K t' \right) \\ y &= y' \\ z &= z' \end{aligned}$$

Using the (x'_s, y'_s, t'_s) coordinates from part (d), we have

$$\boxed{\begin{aligned} x_s &= \gamma_K v_K (\Delta t_1 + \gamma_{esc}\,\Delta t_2) \\ y_s &= \gamma_{esc} v_{esc}\,\Delta t_2 \\ t_s &= \gamma_K (\Delta t_1 + \gamma_{esc}\,\Delta t_2) \ . \end{aligned}}$$

f) Using the four-velocity found in part (c), one has

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{v_y}{v_x}\right) \\ &= \tan^{-1}\left(\frac{u^2}{u^1}\right) \end{aligned}$$

$$\theta = \boxed{\tan^{-1}\left(\frac{v_{esc}}{\gamma_K v_K}\right) \ .}$$

Alternatively, one can look at the motion between the launching of the escape vehicle in part (a), and the emission of the telemetry signal in part (e). Between these two events, the coordinate interval is

$$\Delta x = x_s - x_{esc} = \gamma_K v_K \gamma_{esc}\, \Delta t_2$$
$$\Delta y = y_s = \gamma_{esc} v_{esc}\, \Delta t_2 \ .$$

So,

$$\theta = \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right) = \tan^{-1}\left(\frac{v_{esc}}{\gamma_K v_K}\right) ,$$

in agreement with the previous answer.

GRADE DISTRIBUTION FOR QUIZ 1