MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — March 19, 1996
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QUIZ 1 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

a) The lines were dark, caused by absorption of the radiation in the cooler, outer layers of the sun.

b) Individual stars in the Andromeda Nebula were resolved by Hubble in 1923.

[The other names and dates are not without significance. In 1609 Galileo built his first telescope; during 1609-10 he resolved the individual stars of the Milky Way, and also discovered that the surface of the moon is irregular, that Jupiter has moons of its own, that Saturn has handles (later recognized as rings), that the sun has spots, and that Venus has phases. In 1755 Immanuel Kant published his *Universal Natural History and Theory of the Heavens*, in which he suggested that at least some of the nebulae are galaxies like our own. In 1912 Henrietta Leavitt discovered the relationship between the period and luminosity of Cepheid variable stars. In the 1950s Walter Baade and Allan Sandage recalibrated the extra-galactic distance scale, reducing the accepted value of the Hubble constant by about a factor of 10.]

c)

(i) True. [In 1941, A. McKellar discovered that cyanogen clouds behave as if they are bathed in microwave radiation at a temperature of about 2.3° K, but no connection was made with cosmology.]

(ii) False. [Any radiation reflected by the clouds is far too weak to be detected. It is the bright starlight shining through the cloud that is detectable.]

(iii) True. [Electromagnetic waves at these wavelengths are mostly blocked by the Earth's atmosphere, so they could not be detected directly until high altitude balloons and rockets were introduced into cosmic background radiation research in the 1970s. Precise data was not obtained until the COBE satellite, in 1990.]

(iv) True. [The microwave radiation can boost the CN molecule from its ground state to a low-lying excited state, a state in which the C and N atoms rotate about each other. The population of this low-lying state is therefore determined by the intensity of the microwave radiation. This population is measured by observing the absorption of starlight passing through the clouds, since there are absorption lines in the visible spectrum caused by transitions between the low-lying state and higher energy excited states.]

(v) False. [No chemical reactions are seen.]

d) Aristarchus. [The heliocentric picture was never accepted by other Greek philosophers, however, and was not revived until the publication of *De Revolutionibus Orbium Coelestium* (*On the Revolutions of the Celestial Spheres*) by Copernicus in 1543.]

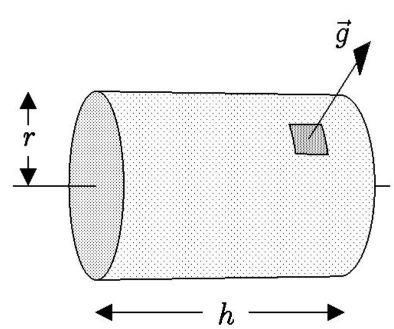
e) (ii) Any patch of the night sky would look as bright as the surface of the sun. [Explanation: The crux of the argument is that the brightness of an object, measured for example by the power per area (i.e., flux) hitting the retina of your eye, does not change as the object is moved further away. The power falls off with the square of the distance, but so does the area of the image on your retina — so the power per area is independent of distance. Under the assumptions stated, your line of sight will eventually hit a star no matter what direction you are looking. The energy flux on your retina will therefore be the same as in the image of the sun, so the entire sky will appear as bright as the surface of the sun.]

PROBLEM 2: A CYLINDRICAL UNIVERSE

a) Gauss's law of gravity states that

$$\oint \vec{g} \cdot d\vec{s} = -4\pi G M \ ,$$

where $\vec{g}$ is the acceleration of gravity, G is Newton's constant, and M is the total mass enclosed inside the volume. Apply this relation to the following cylinder:



By symmetry $\vec{g}$ points radially outward, so the dot product $\vec{g} \cdot d\vec{s}$ vanishes for the disks that bound the cylinder on the left and right. The only contribution comes from the curved surface of the cylinder, for which the cosine of the dot product is 1. Thus,

$$\oint \vec{g} \cdot d\vec{s} = 2\pi r h g_r \ ,$$

where g_r is the radial component of $\vec{g}$. The mass enclosed, M, is the length times the mass per length, or $h\mu$. Therefore

$$2\pi r h g_r = -4\pi G h \mu \ ,$$

so

$$g_r = -\frac{2G\mu}{r} \; .$$

Since the other components of $\vec{g}$ vanish by symmetry,

$$\boxed{\vec{g} = -\frac{2G\mu}{r}\,\hat{r} \; ,}$$

and

$$\boxed{A = 2G} \; .$$

b) $\ddot{r}$ is just the acceleration of r due to gravity, so

$$\ddot{r} = g_r = -\frac{2G\mu}{r} \; ,$$

where I used the answer from the previous part. The mass per length enclosed within a given cylindrical shell does not change with time, as all the shells move together. It can therefore be evaluated at the initial time t_i. For definiteness we can consider a length h of the cylinder, so the volume of the cylinder of radius r_i is $\pi r_i^2 h$. The mass per length is then

$$\mu(r_i) = \frac{\pi r_i^2 h \rho_i}{h} = \pi r_i^2 \rho_i \; .$$

Thus,

$$\boxed{\ddot{r} = -\frac{2\pi G r_i^2 \rho_i}{r} \; .}$$

c) The function $u(r_i, t)$ is determined by the differential equation that it obeys, combined with the initial conditions. Using the answer from (b),

$$\ddot{u} = \frac{\ddot{r}}{r_i} = -\frac{2\pi G r_i \rho_i}{r} = -\frac{2\pi G \rho_i}{u} \; ,$$

so the differential equation does not depend on r_i. Since $r(r_i, t_i) \equiv r_i$, the initial value of u is given by

$$u(r_i, t_i) = 1 \; .$$

Finally, since the initial velocities are set to agree with Hubble's law,

$$\dot{r}(r_i, t_i) = H_i r_i \; ,$$

it follows that

$$\dot{u}(r_i, t_i) = \frac{\dot{r}(r_i, t_i)}{r_i} = H_i \ .$$

Thus, neither the differential equation for $u(r_i, t)$ nor the initial conditions depend on r_i, so the solution will not depend on r_i.

d) For clarity, we can consider a finite length h of the cylinder. The mass contained inside a cylinder of radius r_i at the initial time t_i is then

$$M(r_i, h) = \pi r_i^2 h \rho_i \ .$$

At time t, this same mass will be uniformly spread in a cylinder of radius $r(r_i, t)$ and length h. The density is therefore

$$\rho(t) = \frac{M(r_i, h)}{\pi r^2 h} = \boxed{\frac{\rho_i}{R^2(t)} \ .}$$

Using this result to replace ρ_i in the differential equation found in (c),

$$\boxed{\ddot{R} = -2\pi G \rho R \ .}$$

e) Multiplying the differential equation by $\dot{R}$,

$$\dot{R} \left[\ddot{R} + \frac{2\pi G \rho_i}{R} \right] = 0 \ .$$

Note that I wrote the differential equation in terms of ρ_i rather than ρ, since the time independence of ρ_i allows us to proceed easily. Rewrite the expression as

$$\frac{d}{dt} \left[\frac{1}{2} \dot{R}^2 + 2\pi G \rho_i \ln R \right] = 0 \ .$$

Thus, the quantity in square brackets must be constant:

$$E = \frac{1}{2} \dot{R}^2 + 2\pi G \rho_i \ln R \ ,$$

so

$$\boxed{V(R) = 2\pi G \rho_i \ln R \ .}$$

The potential energy term $V(R)$ grows as $\ln R$ and is hence unbounded. No matter how large the initial value of $\dot{R}^2$, there can never be enough energy to allow the universe to grow to arbitrarily large R. Eventually the $V(R)$ term will grow to be as large as E, at which point $\dot{R}$ will vanish and then change sign. This universe necessarily recollapses.

PROBLEM 3: A FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$

a) In general, the Hubble constant is given by $H = \dot{R}/R$, where the overdot denotes a derivative with respect to cosmic time t. In this case

$$H = \frac{1}{bt^{3/5}}\,\frac{3}{5}\,bt^{-2/5} = \boxed{\frac{3}{5t}\ .}$$

b) In general, the (physical) horizon distance is given by

$$\ell_{p,\text{horizon}}(t) = R(t)\int_0^t \frac{c}{R(t')}dt'\ .$$

In this case one has

$$\ell_{p,\text{horizon}}(t) = bt^{3/5}\int_0^t \frac{c}{bt'^{3/5}}dt' = ct^{3/5}\,\frac{5}{2}\left[t^{2/5} - 0^{2/5}\right] = \boxed{\frac{5}{2}\,ct\ .}$$

c) The coordinate speed of light is $c/R(t)$, so the coordinate distance that light travels between t_A and t_B is given by

$$\ell_c = \int_{t_A}^{t_B} \frac{c}{R(t')}\,dt' = \int_{t_A}^{t_B} \frac{c}{bt'^{3/5}}\,dt' = \boxed{\frac{5c}{2b}\left(t_B^{2/5} - t_A^{2/5}\right)\ .}$$

d) The physical separation is just the scale factor times the coordinate separation, so

$$\ell_p(t_A) = R(t_A)\,\ell_c = \boxed{\frac{5}{2}ct_A\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right]\ .}$$

$$\ell_p(t_B) = R(t_B)\,\ell_c = \boxed{\frac{5}{2}ct_B\left[1 - \left(\frac{t_A}{t_B}\right)^{2/5}\right]\ .}$$

e) Let t_{eq} be the time at which the light pulse is equidistant from the two galaxies. At this time it will have traveled a coordinate distance $\ell_c/2$, where ℓ_c is the answer to part (c). Since the coordinate speed is $c/R(t)$, the time t_{eq} can be found from:

$$\int_{t_A}^{t_{eq}} \frac{c}{R(t')}\, dt' = \frac{1}{2}\ell_c$$

$$\frac{5c}{2b}\left(t_{eq}^{2/5} - t_A^{2/5}\right) = \frac{5c}{4b}\left(t_B^{2/5} - t_A^{2/5}\right)$$

Solving for t_{eq},

$$\boxed{t_{eq} = \left[\frac{t_A^{2/5} + t_B^{2/5}}{2}\right]^{5/2} .}$$

f) According to Hubble's law, the speed is equal to Hubble's constant times the physical distance. By combining the answers to parts (a) and (d), one has

$$\begin{aligned} v &= H(t_A)\,\ell_p(t_A) \\ &= \frac{3}{5t_A}\,\frac{5}{2}ct_A\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right] = \boxed{\frac{3}{2}c\left[\left(\frac{t_B}{t_A}\right)^{2/5} - 1\right] .} \end{aligned}$$

g) The redshift for radiation observed at time t can be written as

$$1 + Z = \frac{R(t)}{R(t_e)} ,$$

where t_e is the time that the radiation was emitted. Solving for t_e,

$$t_e = \frac{t}{(1+Z)^{5/3}} .$$

As found in part (d), the physical distance that the light travels between t_e and t is given by

$$\ell_p = R(t)\int_{t_e}^{t} \frac{c}{R(t')}\, dt' = \frac{5}{2}ct\left[1 - \left(\frac{t_e}{t}\right)^{2/5}\right] .$$

Substituting the expression for t_e, one has

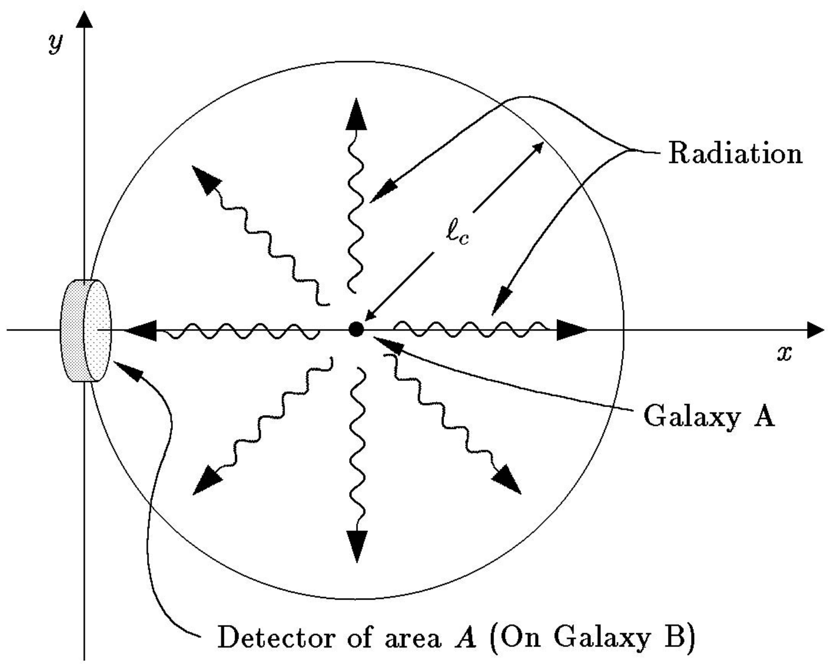
$$\boxed{\ell_p = \frac{5}{2}\,ct\left[1 - \frac{1}{(1+Z)^{2/3}}\right] .}$$

As $Z \to \infty$, this expression approaches

$$\boxed{\lim_{Z\to\infty} \ell_p = \frac{5}{2}\, ct \ ,}$$

which is exactly equal to the horizon distance. It is a general rule that the horizon distance corresponds to infinite redshift Z.

h) Again we will view the problem in comoving coordinates. Put galaxy B at the origin, and galaxy A at a coordinate distance ℓ_c along the x-axis. Draw a sphere of radius ℓ_c, centered galaxy A. Also draw a detector on galaxy B, with physical area A (measured at the present time).



The energy from the quasar will radiate uniformly on the sphere. The detector has a physical area A, so in the comoving coordinate picture its area in square notches would be $A/R(t_B)^2$. The detector therefore occupies a fraction of the sphere given by

$$\frac{[A/R(t_B)^2]}{4\pi\ell_c^2} = \frac{A}{4\pi\ell_p(t_A)^2} \ ,$$

so this fraction of the emitted photons will strike the detector.

Next consider the rate of arrival of the photons at the sphere. In lecture we figured out that if a periodic wave is emitted at time t_A and observed at time t_B, then the rate of arrival of the wave crests will be slower than the rate of emission by a redshift factor $1+Z = R(t_B)/R(t_A)$. The same argument will apply to the rate of arrival of photons, so the rate of photon arrival at the sphere will be slower than the rate of emission by the factor $1+Z$, reducing the energy flux by this factor. In addition, each photon is redshifted in frequency by $1+Z$. Since the energy of each photon is proportional to its frequency, the energy flux is reduced by an additional factor of $1+Z$. Thus, the rate at which energy reaches the detector is

$$\text{Power hitting detector} = \frac{A}{4\pi\ell_p(t_B)^2}\frac{P}{(1+Z)^2} \ .$$

Once again, using $\ell_p(t_B)$ from part (b), we find

$$\boxed{J = \frac{\text{Power hitting detector}}{A} = \frac{P}{25\pi\, c^2\, t_B^2 (1+Z)^{2/3}\left[(1+Z)^{2/3}-1\right]^2} \ .}$$

Alternatively, the answer could be expressed in terms of t_A instead of Z, yielding

$$\boxed{J = \frac{P(t_A/t_B)^{6/5}}{25\pi\, c^2\, t_B^2 \left[1 - \left(\frac{t_A}{t_B}\right)^{2/5}\right]^2} \ .}$$

i) Let t'_A be the time at which the light pulse arrives back at galaxy A. The pulse must therefore travel a coordinate distance ℓ_c (the answer to part (c)) between time t_B and t'_A, so

$$\int_{t_B}^{t'_A} \frac{c}{R(t')}\, dt' = \ell_c \ .$$

Using the answer from (c) and integrating the left-hand side,

$$\frac{5c}{2b}\left(t_A'^{2/5} - t_B^{2/5}\right) = \frac{5c}{2b}\left(t_B^{2/5} - t_A^{2/5}\right) \ .$$

Solving for t'_A,

$$\boxed{t'_A = \left(2t_B^{2/5} - t_A^{2/5}\right)^{5/2} \ .}$$

GRADE DISTRIBUTION FOR QUIZ 1