MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 11, 1994
Prof. Alan Guth

QUIZ 2 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

a) The other two conserved quantities are **baryon number** and **lepton number**. (Weinberg also mentions that the electron lepton number and the muon lepton number appear to be separately conserved. Today we would have to add tau lepton number to this list. These conservation laws are still consistent with all known experiments, but there are theoretical reasons for doubting their exactness. We will talk about this later in the course.)

b) It is approximately 10^{-9}.

c) Photons and neutrinos. (Protons and neutrons do not become an appreciable part of the mass density until about 100,000 years after the big bang.)

d) b: all elements other than hydrogen, helium, and perhaps lithium were synthesized primarily in stars.

e) The weak and electromagnetic interactions. At temperatures above $kT \approx 300$ GeV (1 GeV $\equiv 10^9$ eV), these two interactions are believed to merge into what is often called the electroweak interaction. (It is also speculated, in a class of theories called grand unified theories, that the electroweak interactions merge with the strong interactions at $kT \approx 10^{16}$ GeV.)

PROBLEM 2: AN EXPONENTIALLY EXPANDING UNIVERSE

(a) According to Eq. (3.7), the Hubble constant is related to the scale factor by

$$H = \dot{R}/R \ .$$

So

$$H = \frac{\chi R_0 e^{\chi t}}{R_0 e^{\chi t}} = \boxed{\chi \ .}$$

(b) According to Eq. (3.8), the coordinate velocity of light is given by

$$\frac{dx}{dt} = \frac{c}{R(t)} = \frac{c}{R_0} e^{-\chi t} \ .$$

Integrating,

$$\begin{aligned} x(t) &= \frac{c}{R_0}\int_0^t e^{-\chi t'}\,dt' \\ &= \frac{c}{R_0}\left[-\frac{1}{\chi}e^{-\chi t'}\right]_0^t \\ &= \boxed{\frac{c}{\chi R_0}\left[1-e^{-\chi t}\right]\ .} \end{aligned}$$

(c) From Eq. (3.11), or from the front of the quiz, one has

$$1+Z = \frac{R(t_r)}{R(t_e)}\ .$$

Here $t_e = 0$, so

$$\begin{aligned} 1+Z &= \frac{R_0 e^{\chi t_r}}{R_0} \\ &\Longrightarrow \quad e^{\chi t_r} = 1+Z \\ &\Longrightarrow \quad \boxed{t_r = \frac{1}{\chi}\ln(1+Z)\ .} \end{aligned}$$

(d) The coordinate distance is $x(t_r)$, where $x(t)$ is the function found in part (b), and t_r is the time found in part (c). So

$$e^{\chi t_r} = 1+Z\ ,$$

and

$$\begin{aligned} x(t_r) &= \frac{c}{\chi R_0}\left[1-e^{-\chi t_r}\right] \\ &= \frac{c}{\chi R_0}\left[1-\frac{1}{1+Z}\right] \\ &= \frac{cZ}{\chi R_0(1+Z)}\ . \end{aligned}$$

The physical distance at the time of reception is found by multiplying by the scale factor at the time of reception, so

$$\ell_p(t_r) = R(t_r)x(t_r) = \frac{cZe^{\chi t_r}}{\chi(1+Z)} = \boxed{\frac{cZ}{\chi}\ .}$$

PROBLEM 3: LENGTHS AND AREAS IN A TWO-DIMENSIONAL METRIC

a) Along the first segment $d\theta = 0$, so $ds^2 = (1+ar)^2\, dr^2$, or $ds = (1+ar)\, dr$. Integrating, the length of the first segment is found to be

$$S_1 = \int_0^{r_0} (1 + ar)\, dr = r_0 + \frac{1}{2} a r_0^2 \ .$$

Along the second segment $dr = 0$, so $ds = r(1 + br)\, d\theta$, where $r = r_0$. So the length of the second segment is

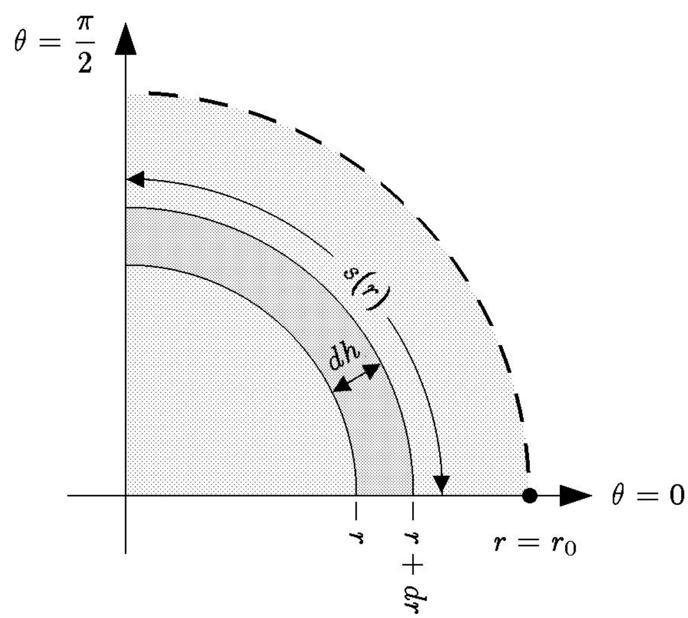
$$S_2 = \int_0^{\pi/2} r_0 (1 + br_0)\, d\theta = \frac{\pi}{2} r_0 (1 + br_0) \ .$$

Finally, the third segment is identical to the first, so $S_3 = S_1$. The total length is then

$$S = 2S_1 + S_2 = 2\left(r_0 + \frac{1}{2} a r_0^2\right) + \frac{\pi}{2} r_0 (1 + br_0)$$

$$= \boxed{\left(2 + \frac{\pi}{2}\right) r_0 + \frac{1}{2}(2a + \pi b) r_0^2 \ .}$$

b) To find the area, it is best to divide the region into concentric strips as shown:



Note that the strip has a coordinate width of dr, but the distance across the width of the strip is determined by the metric to be

$$dh = (1 + ar)\, dr \ .$$

The length of the strip is calculated the same way as S_2 in part (a):

$$s(r) = \frac{\pi}{2} r(1 + br) \ .$$

The area is then

$$dA = s(r)\, dh \ ,$$

so

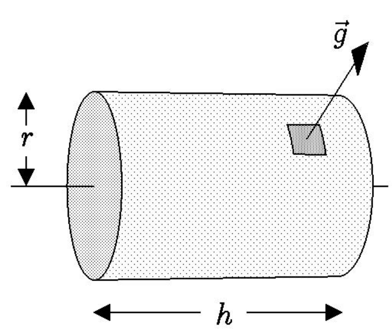
$$\begin{aligned} A &= \int_0^{r_0} s(r)\, dh \\ &= \int_0^{r_0} \frac{\pi}{2} r(1+br)(1+ar)\, dr \\ &= \frac{\pi}{2}\int_0^{r_0} \left[r + (a+b)r^2 + abr^3\right] dr \\ &= \boxed{\frac{\pi}{2}\left[\frac{1}{2}r_0^2 + \frac{1}{3}(a+b)r_0^3 + \frac{1}{4}abr_0^4\right]} \end{aligned}$$

PROBLEM 4: A CYLINDRICAL UNIVERSE

a) Gauss's law of gravity states that

$$\oint \vec{g} \cdot d\vec{s} = -4\pi G M \ ,$$

where $\vec{g}$ is the acceleration of gravity, G is Newton's constant, and M is the total mass enclosed inside the volume. Apply this relation to the following cylinder:



By symmetry $\vec{g}$ points radially outward, so the dot product $\vec{g} \cdot d\vec{s}$ vanishes for the disks that bound the cylinder on the left and right. The only contribution comes from the curved surface of the cylinder, for which the cosine of the dot product is 1. Thus,

$$\oint \vec{g} \cdot d\vec{s} = 2\pi r h g_r \ ,$$

where g_r is the radial component of $\vec{g}$. The mass enclosed, M, is the length times the mass per length, or $h\mu$. Therefore

$$2\pi r h g_r = -4\pi G h \mu \ ,$$

so

$$g_r = -\frac{2G\mu}{r} \ .$$

Since the other components of $\vec{g}$ vanish by symmetry,

$$\boxed{\vec{g} = -\frac{2G\mu}{r}\hat{r} \ ,}$$

and

$$\boxed{A = 2G} \ .$$

b) $\ddot{r}$ is just the acceleration of r due to gravity, so

$$\ddot{r} = g_r = -\frac{2G\mu}{r} \ ,$$

where I used the answer from the previous part. The mass per length enclosed within a given cylindrical shell does not change with time, as all the shells move together. It can therefore be evaluated at the initial time t_i. For definiteness we can consider a length h of the cylinder, so the volume of the cylinder of radius r_i is $\pi r_i^2 h$. The mass per length is then

$$\mu(r_i) = \frac{\pi r_i^2 h \rho_i}{h} = \pi r_i^2 \rho_i \ .$$

Thus,

$$\boxed{\ddot{r} = -\frac{2\pi G r_i^2 \rho_i}{r} \ .}$$

c) The function $u(r_i, t)$ is determined by the differential equation that it obeys, combined with the initial conditions. Using the answer from (b),

$$\ddot{u} = \frac{\ddot{r}}{r_i} = -\frac{2\pi G r_i \rho_i}{r} = -\frac{2\pi G \rho_i}{u} \; ,$$

so the differential equation does not depend on r_i. Since $r(r_i, t_i) \equiv r_i$, the initial value of u is given by

$$u(r_i, t_i) = 1 \; .$$

Finally, since the initial velocities are set to agree with Hubble's law,

$$\dot{r}(r_i, t_i) = H_i r_i \; ,$$

it follows that

$$\dot{u}(r_i, t_i) = \frac{\dot{r}(r_i, t_i)}{r_i} = H_i \; .$$

Thus, neither the differential equation for $u(r_i, t)$ nor the initial conditions depend on r_i, so the solution will not depend on r_i.

d) For clarity, we can consider a finite length h of the cylinder. The mass contained inside a cylinder of radius r_i at the initial time t_i is then

$$M(r_i, h) = \pi r_i^2 h \rho_i \; .$$

At time t, this same mass will be uniformly spread in a cylinder of radius $r(r_i, t)$ and length h. The density is therefore

$$\rho(t) = \frac{M(r_i, h)}{\pi r^2 h} = \boxed{\frac{\rho_i}{R^2(t)} \; .}$$

Using this result to replace ρ_i in the differential equation found in (c),

$$\boxed{\ddot{R} = -2\pi G \rho R \; .}$$

e) Multiplying the differential equation by $\dot{R}$,

$$\dot{R}\left[\ddot{R} + \frac{2\pi G \rho_i}{R}\right] = 0 \; .$$

Note that I wrote the differential equation in terms of ρ_i rather than ρ, since the time independence of ρ_i allows us to proceed easily. Rewrite the expression as

$$\frac{d}{dt}\left[\frac{1}{2}\dot{R}^2 + 2\pi G\rho_i \ln R\right] = 0 \ .$$

Thus, the quantity in square brackets must be constant:

$$E = \frac{1}{2}\dot{R}^2 + 2\pi G\rho_i \ln R \ ,$$

so

$$\boxed{V(R) = 2\pi G\rho_i \ln R \ .}$$

The potential energy term $V(R)$ grows as $\ln R$ and is hence unbounded. No matter how large the initial value of $\dot{R}^2$, there can never be enough energy to allow the universe to grow to arbitrarily large R. Eventually the $V(R)$ term will grow to be as large as E, at which point $\dot{R}$ will vanish and then change sign. This universe necessarily recollapses.

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