MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

PROBLEM SET 4

(Corrected Version)

REFERENCES: Peskin and Schroeder, Chapter 2

NOTE ABOUT CORRECTED VERSION:

The formula at the top of p. 3 has been corrected.

Problem 1: Casimir effect in one dimension

So far we have ignored the zero-point energy of the vacuum $\sum \frac{1}{2}\hbar\omega_k$ as an unobservable (infinite) shift in the zero of the energy scale. However, as Casimir* discovered in 1948, *differences* in vacuum zero-point energies *are* observable. In this problem and the next we will explore the Casimir effect, which is a force between conducting plates that is caused by the change in vacuum energy that results from the boundary conditions imposed by the plates. We will follow a treatment given in *Quantum Field Theory: From Operators to Path Integrals,* by Kerson Huang, pp. 82–88. (A number of the formulas in the book appear to be misprinted, so I have attempted to correct them here.) I recommend trying these problems first without looking at the book, but you may want to look when you are finished to see a plot of real data illustrating the effect. You are allowed to look at the book, but your answer should not be copied from the book.

As a warmup exercise, consider the modes of a free, massless, scalar field in one spatial dimension, confined to a box of length L. We impose the boundary condition that $\phi(x) = 0$ at $x = 0$ and $x = L$, so the allowed terms in the Fourier expansion are $\sin k_n x$, where $k_n = n\pi/L$, with $n = 1, 2, \ldots$. The vacuum energy density will be infinite, as in three-dimensional field theories, so we begin by introducing a cut-off factor $e^{-\omega/\omega_c}$ to render the vacuum energy finite. After computing the relevant energy difference, we will be able to take the limit $\omega_c \to \infty$. The zero-point energy inside the box is then

$$E_0(L) = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n e^{-\omega_n/\omega_c} \ , \qquad \text{where } \omega_n = n\pi/L.$$

(a) Note that $\sum_n n e^{-na}$ can be written as the derivative of a geometric series. Show that

$$E_0(L) = \frac{\pi}{8L}\frac{1}{\sinh^2(\pi/2\omega_c L)} \xrightarrow[\omega_c\to\infty]{} \frac{L\omega_c^2}{2\pi} - \frac{\pi}{24L} + \mathcal{O}(\omega_c^{-2}) \ .$$

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* H.B.G. Casimir, *Proc. Kon. Ned. Akad. Wetenschap* **B51,** 793 (1948).

(b) Now insert two hard-wall partitions in the box centered about the midpoint and separated by distance a, so that the total zero-point energy $E^{\text{total}}(a)$ becomes

$$E^{\text{total}}(a) = E_0(a) + 2E_0\left(\frac{L-a}{2}\right) .$$

The force between the partitions can be found from $\partial E^{\text{total}}(a)/\partial a$. Calculate the force in the limit $\omega_c \to \infty$ and $L \to \infty$, and state whether it is attractive or repulsive.

Problem 2: The Casimir effect in electrodynamics

In this problem we will calculate the force between two metallic plates in the electrodynamic vacuum. Although we have not yet quantized the electromagnetic field, the rule for calculating the zero-point energy is simple: $\frac{1}{2}\hbar\omega$ for each normal mode of oscillation of the classical field theory.

(a) First, calculate the normal modes in a perfectly conducting box of size $a \times b \times c$. Use Coulomb gauge $\nabla \cdot \mathrm{A} = 0$ and $A_0 = 0$. Show that the boundary conditions $E_\parallel = 0$, $B_\perp = 0$ imply the Coulomb gauge boundary conditions $A_\parallel = 0$, $\frac{\partial}{\partial n} A_\perp = 0$ (where $\frac{\partial}{\partial n}$ denotes the normal derivative).

(b) Construct a complete set of normal modes (i.e., eigenfunctions of ∇^2), satisfying the gauge conditions and boundary conditions, in the form

$$\begin{aligned} A_x &= \epsilon_x \cos(k_x x)\sin(k_y y)\sin(k_z z) \\ A_y &= \epsilon_y \sin(k_x x)\cos(k_y y)\sin(k_z z) \\ A_z &= \epsilon_z \sin(k_x x)\sin(k_y y)\cos(k_z z) \ . \end{aligned}$$

(For some reason the mode expansion for a rectangular cavity is not included in standard references such as J.D. Jackson, *Classical Electrodynamics*, or P.M. Morse and H. Feshbach, *Methods of Theoretical Physics, Parts I and II*. They are described in W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, Second Edition.) Show that modes exist for the discrete momenta

$$k_x = \frac{\pi n_x}{a} \ , \quad k_y = \frac{\pi n_y}{b} \ , \quad k_z = \frac{\pi n_z}{c} \ , \quad \text{where } n_i = 0, 1, 2, \dots \ .$$

Show that there are two modes whenever all three n_i are nonzero, and one mode when one of the n_i is zero and the other two are nonzero. If more than one n_i is zero, there are no modes.

Hence, show that the zero-point energy, regulated by a cutoff function $F(k)$, is given by

$$E_0(a,b,c) = \frac{1}{2}\sum_{n_x,n_y=1}^{\infty} \sqrt{k_x^2+k_y^2}\, F\left(\sqrt{k_x^2+k_y^2}\right) + (k_x,k_y \to k_y,k_z)$$

$$+ (k_x,k_y \to k_x,k_z) + \sum_{n_x,n_y,n_z=1}^{\infty} \sqrt{k_x^2+k_y^2+k_z^2} F\left(\sqrt{k_x^2+k_y^2+k_z^2}\right) .$$

We will assume that $F(k)=1$ for k below some cutoff and goes to zero sufficiently rapidly at large k to yield a finite sum.

(c) Now consider a large cubical box of edge L, divided in the z-direction with two conducting plates centered at the midpoint and separated by a small distance a. For large L one can calculate the vacuum energy by treating k_x and k_y as if they are continuous, but k_z must be summed over the allowed values found in the previous part. Defining $k \equiv \sqrt{k_x^2+k_y^2}$, show that

$$E_0(a,L,L) = \frac{L^2}{\pi^2}\int_0^{\infty} \mathrm{d}k_x \mathrm{d}k_y \left[\frac{1}{2}kF(k) + \sum_{n=1}^{\infty}\sqrt{k^2+\frac{\pi^2 n^2}{a^2}}F\left(\sqrt{k^2+\frac{\pi^2n^2}{a^2}}\right)\right]$$

$$= \frac{L^2}{4\pi}\int_0^{\infty}\mathrm{d}k\, k^2\, F(k) + \frac{\pi^2 L^2}{4a^3}\sum_{n=1}^{\infty} G(n) ,$$

where

$$G(n) = \int_{n^2}^{\infty} \mathrm{d}y\, \sqrt{y}\, F\left(\frac{\pi\sqrt{y}}{a}\right) .$$

(d) Use the Euler-MacLauren formula*

$$\sum_{n=1}^{\infty} G(n) = \int_0^{\infty} \mathrm{d}n\, G(n) - \frac{1}{2}G(0) - \frac{B_2}{2!}G'(0) - \frac{B_4}{4!}G'''(0) + \dots ,$$

where B_n denote the Bernoulli numbers, with $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, to show that

$$E_0(a,L,L) = L^2\left[c_1 a + c_2 - \frac{\pi^2}{720 a^3}\right] .$$

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* M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables*, p. 806; E.T. Whittaker and N. Watson, *A Course of Modern Analysis*, pp. 127–128. Here we follow the notation of Abramowitz and Stegun, who use a definition of the Bernoulli numbers that differs from that of Whittaker and Watson.

Show that all derivatives $d^k G(n)/dn^k$, evaluated at $n = 0$, vanish when $k > 3$, so no further terms in the Euler-MacLauren expansion need be considered. Write expressions for c_1 and c_2.

(e) Finally, calculate the total energy

$$E^{\text{total}}(a) = E_0(a, L, L) + 2E_0\left(\frac{L-a}{2}, L, L\right)$$

to show that the attractive pressure (force per unit area) between two capacitor plates is $P = \pi^2/(240a^4)$. Put in the appropriate powers of $\hbar$ and c to show that $P = 0.013/a^4$ dynes/cm^2, where a is in μm (micrometers, 10^{-6} m).