8.324 Test # 1

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Closed notes.

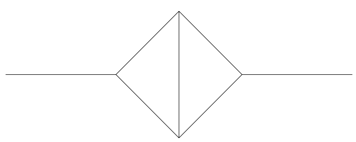
Duration: 90 minutes. Suggested time management: answer the first four questions in 30 minutes or less, leaving 15 minutes or more for each of the last four questions.

Question 1. (10 points) A matrix L is a Lorentz transformation if and only if

$$L^T \eta L = \eta\,,$$

where η is the Minkowski metric. Show that L^T is also a Lorentz transformation.

Question 2. (10 points) Consider the following Feynman graph in a ϕ^3 theory in four spacetime dimensions.



What is the symmetry factor for this graph? What is its degree of divergence D?

Question 3. (10 points) A Lie algebra with generators x (the first generator) and y (the second generator) has commutator $[x, y] = x + y$. Write the matrices ad x and ad y.

Question 4. (10 points) The dimensionally extended euclidean propagator for a massive field is

$$\Delta_E(x) = \int \frac{d^n k}{(2\pi)^n} \frac{e^{ik\cdot x}}{k^2 + m_0^2}\,.$$

Calculate $\Delta_E(0)$ as a function of m_0 and n (recall $\Gamma(s) = \int_0^\infty dt e^{-t} t^{s-1}$).

Question 5. (15 points) Let A, B, and C be three real scalar fields with dynamics governed by

$$\mathcal{L} = -\frac{1}{2}\Big((\partial_\mu A)^2 + m^2 A^2 + (\partial_\mu B)^2 + (\partial_\mu C)^2\Big) - gABC\,.$$

Calculate the decay rate Γ for the A particle to lowest order in the coupling constant.[1]

[1] For this you should recall that for

$$\langle p_1, p_2, \cdots p_n; + | p; -\rangle = -i\,(2\pi)^4\,\delta^4(p - \sum p_f)\,T(p_1 \cdots p_n; p)\,,$$

the decay rate Γ is given by

$$\Gamma = \Big[\prod_{i=1}^{n} \int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2p_i^0}\Big]\,(2\pi)^4\,\delta^4(p - \sum p_f)\,|T(p_1 \cdots p_n; p)|^2\,\frac{1}{2p^0}\,.$$

A factor of $1/k!$ must be added for each group of k identical particles in the final state.

Question 6. (15 points) After a week-long calculation a graduate student has determined the relation between the bare and renormalized couplings λ_0 and λ to two loops. Unfortunately, coffee spills and makes a smudge out of one of the coefficients on the right-hand side:

$$\lambda_0 = \lambda\mu^{4-n}\Big[1 - \lambda\frac{3}{n-4} - \lambda^2\Big(\frac{5}{n-4} - \frac{[illegible]}{(n-4)^2}\Big)\Big]\,.$$

Help the graduate student determine the number that is missing in the formula.

Question 7. (15 points) Consider a theory with $SU(3)$ global symmetry and a set of scalar fields M in the adjoint representation: M is an antihermitian traceless 3-by-3 matrix acted on by $SU(3)$ elements via conjugation. Consider the expectation value $\bar{M} = ia\,\mathrm{diag}(1,1,-2)$, with $a \neq 0$ real.

(a) What is the surviving symmetry? How many Goldstone bosons do you get?

(b) Assume now the theory has *local* $SU(3)$ symmetry. The kinetic terms for M take the form

$$\mathcal{L}_{kin} = -\frac{1}{2}\eta^{\mu\nu}\,\mathrm{tr}\,\big((D_\mu M)^\dagger(D_\nu M)\big)\,, \quad \text{with} \quad D_\mu M = \partial_\mu M - g[A_\mu, M])\,.$$

Assume that the gauge fields

$$A_\mu = \begin{pmatrix} iA_\mu^{11} & A_\mu^{12} & A_\mu^{13} \\ -A_\mu^{12*} & iA_\mu^{22} & A_\mu^{23} \\ -A_\mu^{13*} & -A_\mu^{23*} & iA_\mu^{33} \end{pmatrix}\,, \quad A_\mu^{ii}\ \text{real}, \ \ A_\mu^{ij} = \frac{1}{\sqrt{2}}(B_\mu^{ij} + iC_\mu^{ij})\ \text{complex}\ (i \neq j), \qquad (1)$$

are canonically normalized (also, $A_\mu^{33} = -A_\mu^{11} - A_\mu^{22}$). How many massive vectors are there after $\bar{M}$ induces symmetry breaking? What are the masses of these fields?

Question 8. (15 points) Consider the following action for scalar fields on a three-dimensional spacetime M_3:

$$S = \int_{M_3} d^3x\,\mathcal{L}\,, \quad \mathcal{L} = -\epsilon^{\alpha\beta\gamma}\mathrm{tr}\,\big((g^{-1}\partial_\alpha g)\,(g^{-1}\partial_\beta g)\,(g^{-1}\partial_\gamma g)\big)\,, \quad g = \exp\big(\phi^a t_a\,\big)\,.$$

Here $\alpha, \beta, \gamma = 0, 1, 2$, are spacetime indices and $\epsilon^{\alpha\beta\gamma}$, with $\epsilon^{012} = 1$, is the totally antisymmetric symbol. The ϕ^a are a set of scalar fields and t_a are generators for a Lie algebra presented with totally antisymmetric structure constants f_{abc} and normalization $\mathrm{tr}\,(t_a t_b) = -2\delta_{ab}$.

(a) Calculate the term cubic in fields in $\mathcal{L}$. The answer should have no left-over trace.

(b) Show that the term found in (a) is a total derivative (actually $\mathcal{L}$ is a total derivative!).

(c) Let M_3, with coordinates (x^0, x^1, x^2) and $x^2 \leq 0$, be a space whose boundary $(x^0, x^1, x^2 = 0)$ is the two-dimensional spacetime M_2. Write the term obtained in (a) as an action for two-dimensional scalar fields living on M_2. Use indices i, j for two-dimensional coordinates.