8.324 – Homework 6

Due: Tuesday 15 November 2005

Problem 1 Ideals and two-dimensional Lie algebras.

(a) Exhibit an ideal in the algebra $u(n)$. Conclude that $u(n)$ is not simple.

(b) Show that there is a unique non-abelian Lie algebra with two generators and that it can be presented in the form $[x, y] = x$ (and $[x, x] = [y, y] = 0$), where x and y are the two generators. Is this algebra simple? Is it semisimple? Is it solvable? Exhibit a linear Lie algebra isomorphic to this two-dimensional algebra (use the adjoint representation).

Problem 2 Derivations and automorphisms of Lie algebras.

A derivation δ of a Lie algebra $\mathcal{G}$ is a map $\mathcal{G} \to \mathcal{G}$ that satisfies

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

(a) Show that derivations define a Lie algebra under commutation.

(b) Give a formula for $\delta^n[x, y]$, for n a positive integer.

(c) Define the map $e^\delta : \mathcal{G} \to \mathcal{G}$ by the natural action

$$e^\delta(x) = x + \delta(x) + \frac{1}{2}\delta^2(x) + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n(x)$$

and assume this is well defined on $\mathcal{G}$ (namely, the above infinite series converges). Prove that e^δ is an it automorphism of $\mathcal{G}$.

(d) Show that ad x for any $x \in \mathcal{G}$ is a derivation of $\mathcal{G}$. Is ad x an automorphism of $\mathcal{G}$? If yes, explain. If not, how could you construct one using it?

Problem 3 Exercises with Linear Lie algebras.

(a) Consider the classical Lie algebras $L = A_l, B_l, C_l$ and D_l (over real or complex numbers). It is known that the derived algebra of L is equal to L. Prove this for A_l and C_l.

(b) Show that A_1, B_1, and C_1 are isomorphic Lie algebras.

(c) Is D_2 a simple Lie algebra? Explain.

(d) Let $t(n, R)$ denote the set of upper triangular $n \times n$ matrices. These are the matrices where the elements below the diagonal vanish ($a_{ij} = 0$ for $i < j$). Show these matrices define a Lie algebra. Is it simple ? Is it semisimple ? Is it solvable ?

Problem 4 Killing metric.

(a) Consider the Lie algebra $sl(2, R)$ (the analysis and result are identical for $sl(2, C)$). Take as basis of generators

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Prove that $sl(2, R)$ is simple by showing explicitly that it has no non-trivial ideals. Calculate explicitly the Killing metric and verify it is non-degenerate.

(b) Calculate the Killing metric for $su(2)$. Compare with the result in (iii).

Problem 4 Proving some basic Lie algebra facts.

(i) Prove that the Killing form is invariant under automorphisms of the Lie algebra and is associative (see notes for the precise meaning of such statements.)

(ii) Let $x \in gl(n, F)$ have n *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$ (here F is either R or C). Show that the eigenvalues of ad x are the n^2 constants $\lambda_i - \lambda_j$ (with $1 \leq i, j \leq n$). [Hints: Work in a basis where x is diagonal. To get an idea for the proof try explicitly the case $n = 2$. A piece of useful notation: one defines e_{ij} to be the matrix having 1 in the (i, j) position and zero elsewhere.]

(iii) Show that for any semisimple Lie algebra $\kappa^{ab} T_a T_b$ is a Casimir operator (here $\kappa^{ab} \kappa_{bc} = \delta_c^{\ a}$, with κ_{ab} the Killing metric).

Problem 5 Exercises with Casimir operators.

(a) Problem 15.1 (Peskin-Schroeder), parts (c) and (d) only (Peskin's matrices are hermitian, you can multiply them all by an i to use antihermitian matrices).

(b) Problem 15.2. (Peskin-Schroeder). Use $t_a = -i\sigma_a/2$ as the chosen basis of the fundamental representation.

(c) Consider the Lie algebra $\mathcal{G} = so(n, R)$. Fix conventionally $C(n) = 1/2$ for the defining or fundamental representation. Calculate $C_2(\mathcal{G})$.

Problem 6 Conventions do not affect the physics.

Consider a nonabelian gauge theory based on the simple Lie algebra $\mathcal{G}$ associated to the Lie group G. The matter content is n_f species of Dirac spinors each in the same representation r of the group. The beta function $\beta(g)$ associated to the dimensionless gauge coupling g is known to read (we will derive this result in a couple of months)

$$\beta(g) \equiv \mu \frac{dg}{d\mu} = -\frac{g^3}{(4\pi)^2} \left(\frac{11}{3} C_2(\mathcal{G}) - \frac{4}{3} n_f C(r) \right) \tag{1}$$

where $C_2(\mathcal{G})$ is the quadratic Casimir of the adjoint representation of the simple Lie algebra $\mathcal{G}$, and $C(r)$ is the constant defined by $\mathrm{Tr}(t_a^r t_b^r) = -C(r)\delta_{ab}$.

(a) Consider $G = SU(N)$ and let the representation for the matter be the fundamental. Find for $N = 2, 3$, and 4, the maximum values of n_f that would still give asymptotic freedom.

(b) As we have seen, in setting up the calculation of $C_2(\mathcal{G})$ and $C(r)$ one must choose a conventional normalization for the generators of the algebra (see handout, or Peskin-Schroeder around Eqn.15.95). This choice affects the actual numerical values of the constants $C_2(\mathcal{G})$ and $C(r)$. Nevertheless, at the end of the day the choice must have no effect whatsoever on the physics given in (1). Explain how this comes about.