8.324 – Homework 5

Due: Thursday 3 November 2005

Problem 1 The symmetry group of a non-abelian gauge theory based on the Lie group G.

(a) Explain in your own words what is a *trivial* principal bundle having a base manifold M and fiber equal to a Lie group G.

(b) Explain what a global section of this bundle is. Give examples showing such sections exist.

(c) Define the group of nonabelian gauge theory. To this end say what are the elements of the group and what is group multiplication.

Problem 2 Covariant derivatives and adjoint fields.

Consider a *unitary* matrix group G with corresponding Lie algebra $\mathcal{G}$ and let M be $\mathcal{G}$-valued field transforming in the adjoint representation of a group, namely

$$M \to VMV^\dagger \,, \tag{1}$$

where V is a group element in the defining representation ($V^\dagger = V^{-1}$). The Lie algebra valued gauge field $\mathbf{A}_\mu$ tranforms as

$$\mathbf{A}_\mu \to V\mathbf{A}_\mu V^\dagger - \frac{1}{g} V\partial_\mu V^\dagger \,. \tag{2}$$

(a) Show that

$$D_\mu M \equiv \partial_\mu M - g\,[\,\mathbf{A}_\mu, M\,]$$

is a good covariant derivative, namely $D_\mu M$ transforms just as M.

(b) Consider the usual Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \overline{\psi}\,(i\gamma^\mu D_\mu \; - m)\psi \; .$$

Here the unitary matrix group G has Lie algebra generators satisfying $[T_a, T_b] = f_{abc}\, T_c$, with f_{abc} totally antisymmetric. The matter field ψ transforms in some Lie-algebra representation defined by matrices t^a_r . Derive the field equation for the gauge field.

(c) Explain carefully how this field equation is consistent with the (standard) local gauge transformations $\mathbf{A}_\mu \to V\mathbf{A}_\mu V^\dagger - \cdots$, and $\psi \to V_r\,\psi$, where V_r is the representation of V. For this purpose note that the gauge field contribution to the field equation can be written in terms of a covariant derivative (as in part (a)). For the contribution quadratic on the spinor field it will be useful to recall that the adjoint representation of the group, in the present case, acts via real orthogonal matrices.

Problem 3 Flat connections are locally trivial.

Consider a nonabelian gauge theory based on a unitary matrix group and assume we have a configuration for the gauge field $\mathbf{A}_\mu$ such that the corresponding field strength $\mathbf{F}_{\mu\nu}$ vanishes. Such gauge field, or connection, is said to be a *flat connection.* We want to show that such a flat connection is gauge equivalent to zero; more precisely, we can find a well-defined gauge parameter such that after a gauge transformation the gauge field can be made to vanish on a connected region of spacetime. Show this in the following way:

(a) Write a differential equation for the group element $U^\dagger(x)$ that defines the gauge transformation that gauges away $\mathbf{A}_\mu$. Verify that the local integrability condition $[\partial_\mu, \partial_\nu]U^\dagger(x) = 0$ is satisfied for the present problem.

(b) Choose an origin x_0 and take $U^\dagger(x_0) = \mathbf{1}$. Consider a path $x(s)$ starting from $x_0 = x(s = 0)$. Write an ansatz for $U^\dagger(x(s))$. To verify that $U^\dagger$ solves the differential equation requires consideration of more than one path. Explain why. Conclude that your ansatz solves the problem if the value of U at any point is independent of the path chosen.

(c) Show that $U^\dagger(x)$, constructed by using paths, is in fact path independent (this is somewhat challenging!). Hints: Consider two nearby paths $x^\mu(s)$ and $\tilde{x}^\mu(s) = x^\mu(s) + \delta x^\mu(s)$, both of which start at x_0 and end at x_1. Let $U^\dagger(x(s))$ and $\tilde{U}^\dagger(\tilde{x}(s))$ denote the solutions along the paths and $\delta U^\dagger = \tilde{U}^\dagger(s) - U^\dagger(s)$. Study the differential equation satisfied by $U\delta U^\dagger$.

Problem 4 Non-abelian gauge fields are not fully specified by $\mathbf{F}_{\mu\nu}$.

We learned that a connection $\mathbf{A}_\mu$ whose field strength $\mathbf{F}_{\mu\nu}$ vanishes is locally gauge equivalent to the zero connection locally. We now want to ask another question. Suppose two different connections have the same field strength, are the two connections gauge equivalent locally? In other words, is there a gauge transformation that maps one connection into the other (locally)?

(a) Consider the question in the abelian theory and conclude that in this case the gauge fields are gauge equivalent locally.

For the non-abelian theory connections with the same field strength need *not* be gauge equivalent locally. Therefore, it is sufficient to show an example where this is the case. Here is an example which I learned from R. Jackiw. Consider an $SU(2)$ gauge theory and two connections. The first one lives in the third direction of the Lie algebra, and the only nonvanishing components are

$$\mathbf{A}_x = \frac{1}{2}\, g\, y\, \frac{i\sigma^3}{2}\,, \quad \mathbf{A}_y = -\frac{1}{2}\, g\, x\, \frac{i\sigma^3}{2}\,,$$

where g is the gauge coupling constant. The second gauge field will be of the form

$$\mathbf{A}'_x = \frac{i\sigma^1}{2}\,, \quad \mathbf{A}'_y = -\frac{i\sigma^2}{2}\,,$$

with all other components equal to zero.

(b) Show by explicit computation that both connections lead to the same field strength $\mathbf{F}_{\mu\nu}$ ($= \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - g\,[\mathbf{A}_\mu, \mathbf{A}_\nu]$).

(c) It remains to show that the two connections given above are not gauge equivalent. Prove that this is the case by picking a suitable gauge *covariant* local object and showing that while it vanishes for one connection it does not vanish for the other (why is this sufficient ?)