8.324 – Homework 3

Due: Tuesday 4 October 2005

Problem 1. Physical region in scattering amplitude.

Consider elastic scattering of two identical particles of mass m. Let p_1, p_2 denote the incoming momenta, and p_1', p_2' denote the outgoing momenta. Verify that the relativistic invariants s, t, and u obey $s + t + u = 4m^2$. Do a center of mass analysis of the scattering to prove that

$$s \geq 4m^2, \quad -(s - 4m^2) \leq \, t, u \, \leq 0 \, .$$

These inequalities define the so-called "physical region". Let θ denote the angle formed by the spatial momenta $\vec{p}_1$ and $\vec{p}_1{}'$ in the center of mass frame. Calculate t and u in terms of s and θ. Why can one write any Lorentz-invariant function of p_1, p_2, p_1', p_2' in terms of s and t?

Problem 2. Explicit check of Lorentz invariance (J. Goldstone).

In this problem we construct an explicit proof that the *on-shell* measure of integration is invariant under orthochronous Lorentz transformations:

$$\frac{d^3\vec{p}}{p^0} = \frac{d^3\vec{p}'}{p'^0} \tag{1}$$

For this purpose consider first arbitrary Lorentz tranformations $x'^\mu = L^\mu{}_\nu x^\nu$ and view $L^\mu{}_\nu$ as the element of a matrix L with row index μ and column index ν ($\mu, \nu = 0, 1, 2, 3$, and $i, j = 1, 2, 3$). If the Minkowski metric is represented as the matrix η, a matrix L is a Lorentz transformation if and only if:

$$L^T \, \eta \, L = \eta \, .$$

(a) Prove that L^T and L^{-1} are Lorentz transformations if L is.

(b) Consider a $(1 + 3)$ time–space matrix decomposition of L:

$$L = \begin{pmatrix} \alpha & \vec{b}^{\,t} \\ \vec{c} & M \end{pmatrix} .$$

Here α is a number, $\vec{b}$ and $\vec{c}$ are (column) three-vectors, and M is a three-by-three matrix. Give the matrix decomposition of L^{-1} and write out the identities that follow from $L^{-1}L = 1$. Prove that

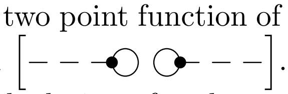
$$\det L = \frac{1}{L^0{}_0} \, \det(L^i{}_j) \, . \tag{2}$$

(c) Prove that the invariance in (1) holds if

$$|\det(L^i{}_j + L^i{}_0 \, v^j)| = L^0{}_0 + L^0{}_i v^i \, . \tag{3}$$

(d) Prove that equation (3) holds (for orthochoronous Lorentz transformations) [Hint: the equation $\det(1 + \vec{u}\,\vec{v}^{\,t}) = 1 + \vec{v}^t \cdot \vec{u}$ is useful.].

Problem 3. Ch.4. Problem 2, p. 217.

Before starting the computation of the graph consider the following issue. To order g_0^2 the naive two point function of the ϕ field contains, among other disconnected vacuum graphs, the graph . Explain why a proper definition of the two point function makes the calculation of such a graph unnecessary.