8.324 – Homework 7

Due: Friday 2 December 2005

Problem 1 Vacua overlap in symmetry breaking.

We showed in lecture that in a scalar field theory with potential $v(\phi) = \frac{\lambda}{4!}(\phi^2 - \phi_+^2)^2$ the overlap between the two vacua $|0+\rangle$ and $|0-\rangle$ of the theory is

$$\langle 0-|0+\rangle = \langle 0|e^{-2i\phi_+\int_V \Pi d^3x}|0\rangle = \exp(-m\phi_+^2 V)\,,$$

where V is the volume of space and $m^2 = \lambda\phi_+^2/3$. Rederive this result using the functional integral to evaluate the matrix element above.

Problem 2 Expectation values of Wilson loops (based on Peskin's 15.3).

Consider a Wilson loop in pure quantum electrodynamics without fermions

$$U_P = \exp\Big(-ie\oint_P dx^\mu A_\mu(x)\Big)\,.$$

Here P is the closed path defining the loop.

(a) Use functional integration to compute the (normalized) exact quantum expectation value $\langle U_P\rangle$. Show that it is given by

$$\langle U_P\rangle = \exp\Big[-\frac{e^2}{8\pi^2}\oint_P dx^\mu \oint_P dy^\nu\, \eta_{\mu\nu}\,\frac{1}{(x-y)^2+i\epsilon}\Big]\,. \tag{1}$$

Hints: Work in the gauge $\partial_\mu A^\mu = 0$. The functional integral may be evaluated directly or using perturbation theory, in which case the two-point function of the gauge field will come handy.

(b) Compute the expectation value for the Wilson loop U_P made of a rectangular path with spacelike width R and timelike length $T \gg R$. Show that the result takes the form

$$\langle U_P\rangle = \exp[-iE(R)T]\,,$$

where $E(R) = -e^2/4\pi R$ is the electrostatic energy of two charges. Give an interpretation of this result by using the gauge $A_0 = 0$ and arguing that the spacelike part of a Wilson line is associated to certain sources.
Hints: There are divergences in the integrals, they correspond to self energy contributions and are to be dropped since they do not depend on the interesting parameters R or T.

(c) Let the nonabelian gauge field two-point function be

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle = \int\frac{d^4p}{(2\pi)^4}\,\frac{-i\eta_{\mu\nu}\delta^{ab}}{p^2}\,e^{-ip\cdot(x-y)}\,.$$

Compute the expectation value of a non-abelian Wilson loop to second order in g (as in equation (1), using an arbitrary loop). The result also depends on the representation of the Lie algebra used for the matrix valued connection. Show that the Coulomb potential of the non-abelian gauge theory is $E(R) = -g^2 C_2(r)/4\pi R$.

Problem 3 Pion Lagrangian and scattering.

Consider the pion lagrangian density

$$\mathcal{L} = -\frac{1}{2}\,\frac{\partial_\mu \vec{\pi}\cdot\partial^\mu \vec{\pi}}{(1+\vec{\pi}^2/F^2)^2}\,,$$

where $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ denotes three real scalar fields assembled into a three component vector.
(a) Calculate the tree level the transition amplitude

$$\langle\, \{p_1', i'\}, \{p_2', j'\}+ \,\big|\, \{p_1, i\}, \{p_2, j\} - \rangle$$

for a process where two incoming pions become two outgoing pions (in the above $i, j\cdots$ are isospin indices).
(b) Calculate the differential scattering cross-section(s) for pions. Assume the incoming and target pions have fixed polarization i and j. Also calculate the cross section for the scattering of the charge eigenstates $\pi^\pm \equiv (\pi_1 \pm i\pi_2)/\sqrt{2}$, $\pi^0 \equiv \pi_3$.
Suggestion: Use the notation, conventions, and results of Brown, Chapter 3.

Problem 4 G and G/H scalar field theories.

(a) Consider a scalar field theory on a group G, with Lie algebra $\mathcal{G}$ (compact, real and semisimple) having totally antisymmetric structure constants f_{abc}. Write out the expression for the kinetic term and that for the interaction term involving the lowest number of scalar fields. To this end, parametrize the group element as $g = \exp(\phi^a t_a/F)$, with $\mathrm{tr}(t_a t_b) = -C\delta_{ab}$ and normalize appropriately the overall lagrangian density so that the resulting quadratic term has standard normalization.

(b) Consider a scalar field theory on $G = SU(2)$. For this write the general group element as

$$g = \begin{pmatrix} \varphi_0 & \varphi_1 \\ -\overline{\varphi}_1 & \overline{\varphi}_0 \end{pmatrix}\,, \quad \text{with} \quad |\varphi_0|^2 + |\varphi_1|^2 = 1\,,$$

where the complex fields φ_0 and φ_1 satisfy the above constraint. Write out the Lagrangian density for the constrained complex fields. Relate this Lagrangian to the $SO(N)/SO(N-1)$ case discussed in class, for the case $N = 4$, and write the action in terms of unconstrained real fields.

(c) To write a theory on $G/H = SU(2)/U(1)$ consider the constrained Lagrangian of (b), and let the $h \in U(1) \subset SU(2)$ act on the group elements $g \in G$ as

$$g \to gh = g\begin{pmatrix} e^{i\theta(x)} & 0 \\ 0 & e^{-i\theta(x)} \end{pmatrix}$$

Write out the $U(1)$ gauge transformations of φ_0 and φ_1, and replace the lagrangian for one with a local $U(1)$ symmetry gauged by an auxiliary, nondynamical gauge field A_μ. Eliminate A_μ by using its equations of motion. Prove that the Lagrangian density is the form (find the exact coefficient!)

$$\mathcal{L} \sim |\overline{\varphi}_1 \partial_\mu \varphi_0 - \varphi_0 \partial_\mu \overline{\varphi}_1|^2$$

Confirm that this Lagrangian has the desired local gauge invariance. Rewrite this Lagrangian in terms of a single complex (gauge invariant) variable $Z = \overline{\varphi}_1/\varphi_0$. Recognize the resulting theory as that of scalars living on S^2.

Problem 5 Breaking $SU(3)$ with an adjoint.

Consider a field theory with an $SU(3)$ global symmetry. The field theory includes a set of scalar fields that transform in the adjoint of $SU(3)$, namely, they can be presented as a matrix $M \in su(3)$ which transforms by conjugation with group elements.

(a) Assume $SU(3)$-invariant potentials $V(M)$ can be designed such that stationary points $M = \bar{M}$ exist where $\bar{M}$ is an arbitrary matrix in $su(3)$. Explain why $\bar{M}$ can be diagonalized by conjugation with an $SU(3)$ matrix. Characterize the possible $SU(3)$-inequivalent vacua $\bar{M}$ and give, for each case, the corresponding unbroken subgroup of $SU(3)$. Is it possible to break $SU(3)$ completely? Explain.

(b) Construct $SU(3)$ invariants I_2 and I_3 quadratic and cubic in M. Explain why invariants based on higher powers of M can be built in terms of I_2 and I_3. How about $\det M$? Consider a general potential $V(I_2, I_3)$ that depends on I_2 and I_3. Derive the equation that determines the stationary points of this potential (watch out! it is easy to miss a term in here.).

(c) Consider the potential $V \sim I_2 + \beta I_3$, with $\beta \neq 0$ an arbitrary constant. Show that there is only one possible pattern of symmetry breaking and it occurs for all $\beta \neq 0$.

(d) Consider the potential $V \sim (I_2 + c)^2$, with c a constant whose sign you may adjust to produce a potential with critical points. Characterize the possible $SU(3)$ inequivalent vacua of this potential and determine the unbroken symmetries on these vacua. What happens to this set of inequivalent vacua when $V \sim (I_2 + c)^2 + I_3$?

Problem 6 Mass matrix for quasi-Goldstone bosons.

We determined that the 8-by-8 mass matrix for quasi-Goldstone bosons arising from breaking of chiral symmetry for three quark flavors takes the form

$$M^2_{ab} = \frac{8v^3}{f^2}\ \mathrm{Tr}\ \left(M\{t^a\,, t^b\}\right),$$

where $M = \mathrm{diag}(m, m, m_s)$ and t^a are a hermitian presentation of the $su(3)$ generators. Calculate the matrix M^2_{ab} and give the spectrum of quasi-Goldstone bosons.

Problem 7 Higgs mechanism in $SU(5)$ symmetry breaking (Peskin 20.1).

Consider a gauge theory with gauge group $SU(5)$ coupled to a scalar field M transforming in the adjoint representation of the group (recall that this means M lives in the Lie algebra $su(5)$). Assume there is a scalar potential inducing symmetry breaking by forcing this scalar field to acquire a vacuum expectation value. Work out the spectrum of gauge bosons and give the unbroken symmetry group for the following two choices of vev:

(a) $M = i\Phi_0 \,\mathrm{diag}\,\{1, 1, 1, 1, -4\}$.

(b) $M = i\Phi_0 \,\mathrm{diag}\,\{2, 2, 2, -3, -3\}$.