8.324 – Homework 4

Due: Tuesday 18 October 2005

Problem 1. Self-energy, cross-sections, and unitarity (based on Brown, Ch.4, Pb.3, p.218.)

This is a scalar field model of the process in which an electron-positron collision produces a quark-antiquark final state through an intermediate photon: $e^+e^- \to q\,\bar{q}$. The role of the electron and that of the positron is played by a massless scalar field ψ. The photon is represented by a massless scalar field ϕ. Finally, the quarks are represented by a massive scalar field χ. The relevant (Minkowskian) Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}(\partial\psi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m^2\chi^2 + \frac{1}{2}g'\phi\psi^2 + \frac{1}{2}g\phi\chi^2 \,.$$

Note that we have introduced two coupling constants. This is only for convenience, at the end we will set $g = g'$.

(a) Consider the tree-level process:

$$\psi\,\psi \to \chi\,\chi$$

which proceeds with an intermediate ϕ. Work in the CM frame. Find the differential cross section $d\sigma/d\Omega$ for the production of a χ particle at the polar angles θ, ϕ, where the z axis coincides with the beam direction.

Suggestions: Once you have calculated the transition amplitude T, to find the differential cross section you should use Eqn.(3.4.22). You can't just use Eqn.(3.4.41) which assumes all masses are the same. Verify that your answer has the correct normalization by taking the limit $m \to 0$ and showing that you get the result in indicated in Eqn.(3.4.41).

(b) Compute the total cross section σ.

(c) Relate σ to the imaginary part of the self-energy $\Sigma(p^2)$ for the ϕ field in Prob. 2 Ch. 4. With m the mass of the χ field, we found

$$\text{Im}\ \Sigma(p^2) = -\frac{g^2}{32\pi}\sqrt{1 + \frac{4m^2}{p^2}}\,, \quad -p^2 \geq 4m^2\,, \tag{1}$$

Suggestions: Apply the unitarity condition $T - T^\dagger = -iT^\dagger T$ to *forward* elastic scattering $\psi\psi \to \psi\psi$, namely $\langle\psi; p_1p_2, +|\psi; p_1p_2, -\rangle$. Find a process whose imaginary part is simply related to the self-energy above. In addition, find out how is σ related to $-iT^\dagger T$.

Problem 2. Asymptotic freedom in six-dimensional field theory.

Consider the field theory

$$\mathcal{L} = -\frac{1}{2}\,(\partial_\mu\phi)^2 - \frac{1}{2}\,m_0^2\phi^2 - \frac{1}{3!}\,g_0\,\phi^3\,,$$

defined as a dimensionally extended Lagrangian to be used in a six-dimensional spacetime.

(a) Compute the one-loop correction to the propagator. Renormalize the mass and find the wave function renormalization factor z_1.

(b) Compute the one-loop correction to the proper vertex $g_0\phi^3$ (the proper vertex is the vertex without propagators attached on the external lines). You will find that the integrals can be done using essentially the same methods discussed in Section 3.5.

(c) Use the results of parts (i) and (ii) to calculate the β-function associated to the coupling constant g. You will find that this is a theory where the coupling constant becomes weaker at higher energies *i.e.* it is an asymptotically free theory.

Problem 3. Renormalization group properties. (Based on Ch.5. Problem 4, p. 280.)

(a) Consider a coupling constant λ and a redefined coupling constant $\bar{\lambda}(\lambda)$. Find the general transformation law for the beta function, namely the relation between $\beta(\lambda)$ and $\overline{\beta}(\overline{\lambda})$. If we think of λ as a coordinate we see that β transforms as a tensor. What kind of tensor ?

(b) Assume that

$$\beta(\lambda) = b_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4 + \cdots$$

and consider the perturbatively defined and invertible coupling constant redefinition:

$$\bar{\lambda}(\lambda) = \lambda + a_2\lambda^2 + a_3\lambda^3 + \cdots .$$

Calculate $\bar{\beta}(\bar{\lambda})$ writing it in the form

$$\bar{\beta}(\bar{\lambda}) = \bar{b}_2\bar{\lambda}^2 + \bar{b}_3\bar{\lambda}^3 + \bar{b}_4\bar{\lambda}^4 + \cdots$$

Verify that, remarkably, $\bar{b}_2 = b_2$ and $\bar{b}_3 = b_3$. What is $\bar{b}_4$? Let $\lambda = \lambda_F$ denote a fixed point. Show that $\bar{\lambda} = \bar{\lambda}_F$ is also a fixed point. How are the derivatives β' and $\bar{\beta}'$ related at the fixed point?

(c) Consider the differential equation

$$\mu\frac{dg^2}{d\mu} = -b(g^2)^2 - c(g^2)^3 - d(g^2)^4 - \cdots \tag{1}$$

and show that one can write a solution of the form

$$\ln\mu = \text{constant} + \frac{1}{bg^2(\mu)} + \frac{c}{b^2}\ln bg^2(\mu) + \mathcal{O}(g^2(\mu)) . \tag{2}$$

Why is the constant of integration above renormalization group invariant? Verify that when c and higher order coefficients in (1) vanish, the constant of integration in (2) is precisely $(\ln\Lambda)$, as given in (5.4.28). We can therefore take the constant to define $(\ln\Lambda)$ in the general case.

(d) Find the first two terms in the asymptotic expansion of $1/g^2(\mu)$ for large μ, in terms of μ and Λ. What is the order of the leading term we are neglecting in this expansion ?

Problem 4. Exercises with degree of divergence.

(a) Use Eqn.(5.2.8) for the degree of divergence to explain why (for the restricted class of theories where this formula is valid) a dimensionless coupling constant implies a renormalizable theory, a coupling constant of positive dimension implies a superrenormalizable theory, and a coupling constant of negative dimension implies a nonrenormalizable theory. [Recall that a renormalizable theory is one where a finite number of amplitudes diverge superficially, this happening at all orders in perturbation theory. A superrenormalizable theory is one where only a finite number of Feynman graphs superficially diverge. A non-renormalizable theory is one where all amplitudes diverge for sufficiently high order in perturbation theory.]

(b) Consider the four-dimensional theory described by the Euclidean Lagrangian

$$-\mathcal{L}_E = -\frac{1}{2}\,(\partial_\mu \phi)^2 - \frac{1}{2}\,m_0^2\,\phi^2 - \frac{1}{3!} g_0\,\phi^3 - c_0\,\phi$$

Write a formula for the (superficial) degree of divergence in terms of the number N of external legs, and the numbers V_1 and V_3 denoting the number of one-point and three-point vertices. List all (connected) superficially divergent graphs for $N = 0, 1, 2, 3$. Include in your list one particle reducible graphs (Partial answer: we found nine graphs for $N = 0$).

(c) Consider QED in four dimensions. Here N_e will denote the number of external electron lines, N_γ denotes the number of external photon lines, P_e denotes the number of electron propagators, P_γ denotes the number of photon propagators, L denotes the number of loop integrals, and V denotes the number of vertices. Each vertex couples two electron lines to a photon line. Moreover, we can assume that the photon propagator carries a $1/p^2$ while the electron propagator carries a $1/p$. Write a formula for the degree of divergence D of a Feynman graph. Show that

$$D = 4 - \frac{3}{2} N_e - N_\gamma$$

List all the superficially divergent Green's functions. As it turns out in QED not every superficially divergent Green's function is actually divergent.

Problem 5. Decoupling a massive field (Based on Ch.5. Problem 5, p. 280.)

Consider the following Lagrangian for two scalar fields ϕ_1 and ϕ_2

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2} m_{1,0}^2 \phi_1^2 - \frac{1}{2} m_{2,0}^2 \phi_2^2 - \frac{\lambda_{11,0}}{4!}\,\phi_1^4 - \frac{\lambda_{12,0}}{2\cdot 2}\,\phi_1^2\phi_2^2 - \frac{\lambda_{22,0}}{4!}\,\phi_2^4\,.$$

(a) Renormalize the general interactions writing some relation between the bare parameters $\lambda_{11,0}, \lambda_{22,0}, \lambda_{12,0}$ and the renormalized parameters $\lambda_{11}, \lambda_{22}, \lambda_{12}$. Use dimensional regularization and minimal subtraction. Find the beta functions.

(b) Suppose we took $\lambda_{11,0} = \lambda_{22,0} = \lambda_0$ and $\lambda_{12,0} = \lambda_0/3$ in the original Lagrangian, so that

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2} m_{1,0}^2 \phi_1^2 - \frac{1}{2} m_{2,0}^2 \phi_2^2 - \frac{\lambda_0}{4!}\,(\phi_1^2 + \phi_2^2)^2\,. \tag{1}$$

Show that the relations between bare and renormalized parameters, as well as the beta functions in (a) are consistent with having a well-defined renormalization and beta function for the single coupling λ_0. Determine this beta function and use it to write $\lambda(\mu)$ in terms of $\lambda(M)$ as in (3.5.47).

The statement of decoupling is the following: In the limit where the mass m_2 becomes much larger than external momenta and the fixed mass unit μ, the field ϕ_2 decouples from the theory defined by (1) and one gets the Lagrangian:

$$\bar{\mathcal{L}} = -\frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}\bar{m}_0^2 \phi_1^2 - \frac{\bar{\lambda}_0}{4!}\,\phi_1^4\,. \tag{2}$$

The renormalized running coupling $\overline{\lambda}(\mu)$ turns out to be different from $\lambda(\mu)$. In addition, $\bar{m}^2(\mu)$ is also different from $m_1^2(\mu)$.

(c) Verify the above claim by computing $\phi_1 - \phi_1$ elastic scattering in the original theory (1) up to one-loop. Take the limit of the amplitude as m_2^2 becomes large (the dispersion relation (3.5.61) comes handy) and prove that the scattering amplitude from $\bar{\mathcal{L}}$ is obtained if, to order λ^2, one has

$$\overline{\lambda}(\mu) = \lambda(\mu) + \frac{1}{6}\lambda^2(\mu)\frac{1}{(4\pi)^2}\ln\frac{m_2^2 e^{\gamma}}{4\pi\mu^2}\,. \tag{3}$$

Argue that this equation implies that

$$\overline{\lambda}(m_2) = \lambda(m_2) + \mathcal{O}(\lambda^2)\,, \tag{4}$$

where $\mathcal{O}(\lambda^2)$ has no large logarithms. Equation (4) means that the running coupling of the original theory and that of the effective theory must be adjusted to match at the mass scale m_2.

(d) Verify the consistency of (3) with the values of the beta functions for λ and for $\overline{\lambda}$.