8.324 – Homework 2

Due: Tuesday 27 September 2005

Problem 1. Unitarity of the S-matrix.

In the scattering of two spin-zero particles we had

$$\begin{aligned}\langle p'q' + |pq-\rangle &= \langle p'|p\rangle\langle q'|q\rangle + \langle p'|q\rangle\langle q'|p\rangle \\ &\quad - i(2\pi)^4\delta^{(4)}(p'+q'-p-q)\,T(p',q';p,q)\end{aligned} \tag{1}$$

It is useful to represent $T(p',q';p,q)$ as a matrix T acting on the state space of 2-particle states of the *free* theory (other particle states are not relevant for total energies that can only lead to elastic scattering). This state space is spanned by states $|pq\rangle = |qp\rangle$, with inner product $\langle p'q'|pq\rangle = \langle p'|p\rangle\langle q'|q\rangle + \langle p'|q\rangle\langle q'|p\rangle$, where $\langle p'|p\rangle = (2\pi)^3 2p^0\delta^{(3)}(\vec{p}-\vec{p}')$. Moreover, the identity matrix is

$$\mathbf{1} = \frac{1}{2}\int d_Lp\, d_Lq\ |pq\rangle\langle pq|, \quad \text{with} \quad d_Lp = \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2p^0}\,.$$

We thus define T as

$$\langle p'q'|T|pq\rangle \equiv (2\pi)^4\,\delta^{(4)}(p'+q'-p-q)\ T(p',q';p,q)\,. \tag{2}$$

Similarly, we define an S-matrix S whose matrix elements between two-particle states given by the transition amplitudes

$$\langle p'q'|S|pq\rangle \equiv \langle p'q' + |pq-\rangle\,. \tag{3}$$

Both the 'in' states $|pq-\rangle$ and the 'out' states $|pq+\rangle$ are complete and normalized like the particle states of the free theory.

(i) Verify that, by definition, $S = 1 - iT$. Explain how $S|pq\rangle$ encodes the future asymptotic state that evolves from the past asymptotic state $|pq-\rangle$.

(ii) Show that $SS^\dagger = S^\dagger S = 1$ and conclude that

$$T - T^\dagger = -iT^\dagger T = -iTT^\dagger\,. \tag{4}$$

Problem 2. Unitarity constraint in ϕ^4 theory.

Verify explicitly the unitarity constraint

$$T - T^\dagger = -iT^\dagger T = -iTT^\dagger$$

for the ϕ^4 theory up to order λ^2. The left hand side follows readily from (3.5.63). The right hand side you must compute.

Problem 3. Two exercises on RG equations.

(a) Equation (3.5.46) in Brown was derived from the differential equation called the renormalization group equation. Verify that you can obtain this result from first principles, namely, that the pairs $(\mu_1, \lambda(\mu_1))$ and $(\mu_0, \lambda(\mu_0))$ yield the same value for λ_0 in Eqn. (3.5.38).

(b) Consider again the scattering of two particles in the CM in the $s \to \infty$ limit, with t/s and u/s finite. Using RG arguments the value of T in this limit was found to be given by Eqn. (3.5.51). On the other hand Eqn.(3.5.66) gives the exact value of $F_{\text{ren}}(s)$. Take the high energy limit of this result, and confirm that (3.5.51) holds precisely. What is the value of the finite term indicated in (3.5.51) ?

Problem 4 . Exercises in dimensional regularization.
Calculate the $n = 4$ pole structure of the following three Euclidean integrals

$$\int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2(p+k)^2} \, ; \quad \int \frac{d^n p}{(2\pi)^n} \frac{p_\mu}{p^2(p+k)^2} \, ; \quad \int \frac{d^n p}{(2\pi)^n} \frac{p_\mu p_\nu}{p^2(p+k)^2}$$

Problem 5. Ch.3. Problem 18, p.189.
In part (a) note that the nonrelativistic limit for the free theory was obtained in (3.1.28). You must only determine the interaction part. To read the potential use (2.3.19) and (2.3.20).

In part (b), to compare with the quantum mechanical potential scattering one must be careful to use the reduced mass and to take into account that the particles are identical.