IV. Perturbative Renormalization Group

IV.A Expectation Values in the Gaussian Model

Can we treat the Landau–Ginzburg Hamiltonian, as a perturbation to the Gaussian model? In particular, for zero magnetic field, we shall examine

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U} \equiv \int d^d\mathbf{x}\left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots\right] + u\int d^d\mathbf{x}\, m^4 + \cdots. \quad \text{(IV.1)}$$

The *unperturbed* Gaussian Hamiltonian can be decomposed into independent Fourier modes, as

$$\beta\mathcal{H}_0 = \frac{1}{V}\sum_{\mathbf{q}} \frac{t + Kq^2 + Lq^4 + \cdots}{2}|m(\mathbf{q})|^2 \equiv \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{t + Kq^2 + Lq^4 + \cdots}{2}|m(\mathbf{q})|^2. \quad \text{(IV.2)}$$

The *interaction* mixes up the normal modes, and

$$\begin{aligned}\mathcal{U} &= u\int d^d\mathbf{x}\, m(\mathbf{x})^4 + \cdots \\ &= u\int d^d\mathbf{x}\int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3 d^d\mathbf{q}_4}{(2\pi)^{4d}} e^{-i\mathbf{x}\cdot(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)} m_\alpha(\mathbf{q}_1)m_\alpha(\mathbf{q}_2)m_\beta(\mathbf{q}_3)m_\beta(\mathbf{q}_4) \\ &\quad + \cdots,\end{aligned} \quad \text{(IV.3)}$$

where summation over α and β is implicit. The integral over $\mathbf{x}$ sets $\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4 = \mathbf{0}$, and

$$\mathcal{U} = u\int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}} m_\alpha(\mathbf{q}_1)m_\alpha(\mathbf{q}_2)m_\beta(\mathbf{q}_3)m_\beta(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) + \cdots. \quad \text{(IV.4)}$$

From the variance of the Gaussian weights, the two–point expectation values in a finite sized system with discretized modes are easily obtained as

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 = \frac{\delta_{\mathbf{q},-\mathbf{q}'}\,\delta_{\alpha,\beta}\, V}{t + Kq^2 + Lq^4 + \cdots}. \quad \text{(IV.5)}$$

In the limit of infinite size, the spectrum becomes continuous, and eq.(IV.5) goes over to

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 = \frac{\delta_{\alpha,\beta}(2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}')}{t + Kq^2 + Lq^4 + \cdots}. \quad \text{(IV.6)}$$

The subscript 0 is used to indicate that the expectation values are taken with respect to the unperturbed (Gaussian) Hamiltonian. Expectation values involving any product of m's can be obtained starting from the identity

$$\left\langle \exp\left[\sum_i a_i m_i\right]\right\rangle_0 = \exp\left[\sum_{i,j} \frac{a_i a_j}{2} \left\langle m_i m_j \right\rangle_0\right], \tag{IV.7}$$

which is valid for any set of Gaussian distributed variables $\{m_i\}$. (This is easily seen by 'completing the square.') Expanding both sides of the equation in powers of $\{a_i\}$ leads to

$$\begin{aligned}&1 + a_i\langle m_i\rangle_0 + \frac{a_i a_j}{2}\langle m_i m_j\rangle_0 + \frac{a_i a_j a_k}{6}\langle m_i m_j m_k\rangle_0 + \frac{a_i a_j a_k a_l}{24}\langle m_i m_j m_k m_k\rangle_0 + \cdots = \\ &1 + \frac{a_i a_j}{2}\langle m_i m_j\rangle_0 + \frac{a_i a_j a_k a_l}{24}\left(\langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0\right) \\ &\quad + \cdots .\end{aligned} \tag{IV.8}$$

Matching powers of $\{a_i\}$ on the two sides of the above equation gives

$$\left\langle \prod_{i=1}^{\ell} m_i \right\rangle_0 = \begin{cases} 0 & \text{for } \ell \text{ odd} \\ \text{sum over all pairwise contractions} & \text{for } \ell \text{ even} \end{cases} . \tag{IV.9}$$

This result is known as *Wick's theorem*; and for example,

$$\langle m_i m_j m_k m_l\rangle_0 = \langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0 .$$

IV.B Expectation values in Perturbation Theory

In the presence of an interaction $\mathcal{U}$, the expectation value of any operator $\mathcal{O}$, is computed perturbatively as follows:

$$\begin{aligned}\langle\mathcal{O}\rangle =&\frac{\int \mathcal{D}\vec{m}\,\mathcal{O}e^{-\beta\mathcal{H}_0-\mathcal{U}}}{\int \mathcal{D}\vec{m}\,e^{-\beta\mathcal{H}_0-\mathcal{U}}} = \frac{\int \mathcal{D}\vec{m}\,e^{-\beta\mathcal{H}_0}\mathcal{O}[1-\mathcal{U}+\mathcal{U}^2/2-\cdots]}{\int \mathcal{D}\vec{m}\,e^{-\beta\mathcal{H}_0}[1-\mathcal{U}+\mathcal{U}^2/2-\cdots]} \\ =&\frac{Z_0[\langle\mathcal{O}\rangle_0 - \langle\mathcal{O}\mathcal{U}\rangle_0 + \langle\mathcal{O}\mathcal{U}^2\rangle_0/2-\cdots]}{Z_0[1-\langle\mathcal{U}\rangle_0 + \langle\mathcal{U}^2\rangle_0/2-\cdots]} .\end{aligned} \tag{IV.10}$$

Inverting the denominator by an expansion in powers of $\mathcal{U}$ gives

$$\begin{aligned}\langle\mathcal{O}\rangle =& \left[\langle\mathcal{O}\rangle_0 - \langle\mathcal{O}\mathcal{U}\rangle_0 + \frac{1}{2}\langle\mathcal{O}\mathcal{U}^2\rangle_0 - \cdots\right]\left[1 + \langle\mathcal{U}\rangle_0 + \langle\mathcal{U}\rangle_0^2 - \frac{1}{2}\langle\mathcal{U}^2\rangle_0 - \cdots\right] \\ =&\langle\mathcal{O}\rangle_0 - \left(\langle\mathcal{O}\mathcal{U}\rangle_0 - \langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0\right) + \frac{1}{2}\left(\langle\mathcal{O}\mathcal{U}^2\rangle_0 - 2\langle\mathcal{O}\mathcal{U}\rangle_0\langle\mathcal{U}\rangle_0 + 2\langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0^2 - \langle\mathcal{O}\rangle_0\langle\mathcal{U}^2\rangle_0\right) \\ &+ \cdots \equiv \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\langle\mathcal{O}\mathcal{U}^n\right\rangle_0^c .\end{aligned} \tag{IV.11}$$

The *connected averages* are defined as the combination of unperturbed expectation values appearing at various orders in the expansion. Their significance will become apparent in diagrammatic representations, and from the following example.

Let us calculate the two point correlation function of the Landau–Ginzburg model to first order in the parameter u. [In view of their expected irrelevance, we shall ignore higher order interactions, and also only keep the lowest order Gaussian terms.] Substituting eq.(IV.4) into eq.(IV.11) yields

$$\begin{aligned}\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 - u\int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}} \\ [\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}') m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0 - \\ \langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 \langle m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0] + \mathcal{O}(u^2).\end{aligned} \tag{IV.12}$$

To calculate $\langle \mathcal{O}\mathcal{U}\rangle_0$ we need the unperturbed expectation value of the product of six m's. This can be evaluated using eq.(IV.9) as the sum of all pair-wise contractions, 15 in all. Three contractions are obtained by first pairing m_α to m_β, and then the remaining four m's in $\mathcal{U}$. Clearly these contractions cancel exactly with corresponding ones in $\langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0$. The only surviving terms involve contractions that connect $\mathcal{O}$ to $\mathcal{U}$. This cancellation persists at all orders, and $\langle \mathcal{O}\mathcal{U}^n\rangle_0^c$ contains only terms in which all $n+1$ operators are connected by contractions. The remaining 12 pairings in $\langle \mathcal{O}\mathcal{U}\rangle_0$ fall into two classes:

(i) 4 pairings involve contracting m_α and m_β to m's with the same index, e.g.

$$\begin{aligned}\langle m_\alpha(\mathbf{q})\, m_i(\mathbf{q}_1)\rangle_0 \langle m_\beta(\mathbf{q}')\, m_i(\mathbf{q}_2)\rangle_0 \langle m_j(\mathbf{q}_3)\, m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0 \\ = \frac{\delta_{\alpha i}\,\delta_{\beta i}\,\delta_{jj}\,(2\pi)^{3d}\,\delta^d(\mathbf{q}+\mathbf{q}_1)\,\delta^d(\mathbf{q}'+\mathbf{q}_2)\,\delta^d(\mathbf{q}_1+\mathbf{q}_2)}{(t+Kq^2)\,(t+Kq'^2)\,(t+Kq_3^2)^2},\end{aligned} \tag{IV.13}$$

where we have used eq.(IV.6). After summing over i and j, and integrating over $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$, such terms make a contribution

$$-4u\frac{n\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{(t+Kq^2)^2}\int\frac{d^d\mathbf{q}_3}{(2\pi)^d}\frac{1}{t+Kq_3^2}. \tag{IV.14}$$

(ii) 8 pairings involve contracting m_α and m_β to m's with different indices, e.g.

$$\begin{aligned}\langle m_\alpha(\mathbf{q})\, m_i(\mathbf{q}_1)\rangle_0 \langle m_\beta(\mathbf{q}')\, m_j(\mathbf{q}_3)\rangle_0 \langle m_i(\mathbf{q}_2)\, m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0 \\ = \frac{\delta_{\alpha i}\,\delta_{\beta j}\,\delta_{ij}\,(2\pi)^{3d}\,\delta^d(\mathbf{q}+\mathbf{q}_1)\,\delta^d(\mathbf{q}'+\mathbf{q}_3)\,\delta^d(\mathbf{q}_1+\mathbf{q}_3)}{(t+Kq^2)\,(t+Kq'^2)\,(t+Kq_2^2)^2}.\end{aligned} \tag{IV.15}$$

Summing over all indices, and integrating over the momenta leads to an overall contribution of

$$-8u\frac{\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{(t+Kq^2)^2}\int\frac{d^d\mathbf{q}_2}{(2\pi)^d}\frac{1}{t+Kq_2^2}. \tag{IV.16}$$

Adding up both contributions, we obtain

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \frac{\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{t+Kq^2}\left[1-\frac{4u(n+2)}{t+Kq^2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}+\mathcal{O}(u^2)\right]. \tag{IV.17}$$

IV.C Diagrammatic Representation of Perturbation Theory

The calculations become more involved at higher orders in perturbation theory. A diagrammatic representation can be introduced to help keep track of all possible contractions. To calculate the ℓ–point expectation value $\langle\prod_{i=1}^{\ell} m_{\alpha_i}(\mathbf{q}_i)\rangle$, at p^{th} order in u, proceed according to the following rules:

(1) Draw ℓ *external points* labelled by $(\mathbf{q}_i, \alpha_i)$ corresponding to the coordinates of the required correlation function. Draw p *vertices* with 4 legs each, labelled by *internal* momenta and indices, e.g. $\{(\mathbf{k}_1, i), (\mathbf{k}_2, i), (\mathbf{k}_3, j), (\mathbf{k}_4, j)\}$. Since the four legs are not equivalent, the four point vertex is indicated by two solid branches joined by a dotted line. (The extension to higher order interactions is straightforward.)

(2) Each point of the graph now corresponds to one factor of $m_{\alpha_i}(\mathbf{q}_i)$, and the unperturbed average of the product is computed by Wick's theorem. This is implemented by joining all external and internal points *pairwise*, by lines connecting one point to another, in all topologically distinct ways (see #5 below).

(3) The algebraic value of each such graph is obtained as follows: **(i)** A line joining a pair of points represents the two point average†; e.g. a connection $(\mathbf{q}, \alpha) \longleftrightarrow (\mathbf{q}', \beta)$, contributes $\delta_{\alpha\alpha'}(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')/(t+Kq^2)$; **(ii)** A vertex is represented by a term such as $u(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$ (the delta-function insures that momentum is conserved).

(4) Integrate over the $4p$ internal momenta $\{\mathbf{k}_i\}$, and sum over the $2p$ internal indices. Note that each closed loop produces a factor of $\delta_{ii} = n$ at this stage.

(5) There is a numerical factor of

$$\frac{(-1)^p}{p!} \times \text{number of different pairings leading to the same topology.}$$

The first contribution comes from the expansion of the exponential; the second merely states that graphs related by symmetry give the same result, and can be calculated once.

(6) When calculating cumulants, only fully connected diagrams (without disjoint pieces) need to be included. This is a tremendous simplification.

† Because of its original formulation in quantum field theory, the line joining two points is usually called a *propagator*. In this context, the line represents the world-line of a particle in time, while the perturbation $\mathcal{U}$ is an 'interaction' between particles. For the same reason, the Fourier index is called a 'momentum.'

IV.D Susceptibility

It is no accident that the correction term in eq.(IV.17) is similar in form to the unperturbed value. This is because the form of the two point correlation function is constrained by symmetries, as can be seen from

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \int d^d\mathbf{x} \int d^d\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{q}'\cdot\mathbf{x}'} \langle m_\alpha(\mathbf{x})\, m_\beta(\mathbf{x}')\rangle. \tag{IV.18}$$

The two point correlation function in real space must satisfy translation and rotation symmetry (in the high temperature phase), and $\langle m_\alpha(\mathbf{x})\, m_\beta(\mathbf{x}')\rangle = \delta_{\alpha\beta}\langle m_1(\mathbf{x}-\mathbf{x}')\, m_1(\mathbf{0})\rangle$. Transforming to center of mass and relative coordinates, the above integral becomes,

$$\begin{aligned}&\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \\ &\int d^d\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) d^d\left(\mathbf{x}-\mathbf{x}'\right) e^{i(\mathbf{q}+\mathbf{q}')\cdot(\mathbf{x}+\mathbf{x}')/2} e^{i(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{q}-\mathbf{q}')/2} \delta_{\alpha\beta}\langle m_1(\mathbf{x}-\mathbf{x}')\, m_1(\mathbf{0})\rangle \\ &\qquad\qquad \equiv (2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')\delta_{\alpha\beta} S(q),\end{aligned} \tag{IV.19}$$

where

$$S(q) = \left\langle |m_1(\mathbf{q})|^2\right\rangle = \int d^d\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} \langle m_1(\mathbf{x}-\mathbf{x}')\, m_1(\mathbf{0})\rangle, \tag{IV.20}$$

is the quantity observed in scattering experiments (sec.II.D).

From eq.(IV.17) we obtain

$$S(q) = \frac{1}{t+Kq^2}\left[1 - \frac{4u\,(n+2)}{t+Kq^2}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2} + \mathcal{O}(u^2)\right]. \tag{IV.21}$$

It is useful to examine the expansion of the inverse quantity

$$S(q)^{-1} = t + Kq^2 + 4u(n+2)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2} + \mathcal{O}(u^2). \tag{IV.22}$$

In the high temperature phase, eq.(IV.20) indicates that the $q \to 0$ limit of S(q) is just the magnetic susceptibility χ. For this reason, $S(q)$ is sometimes denoted by $\chi(q)$. From eq.(IV.22), the inverse susceptibility is given by

$$\chi^{-1}(t) = t + 4u(n+2)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2} + \mathcal{O}(u^2). \tag{IV.23}$$

The susceptibility no longer diverges at $t = 0$, since

$$\chi^{-1}(0) = 4u(n+2)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{Kk^2} = \frac{4(n+2)u}{K}\frac{S_d}{(2\pi)^d}\int_0^\Lambda dk\, k^{d-3} = \frac{4(n+2)u}{K}K_d\left(\frac{\Lambda^{d-2}}{d-2}\right), \tag{IV.24}$$

is a finite number. This is because in the presence of u the critical temperature is reduced to a negative value. The modified critical point is obtained by requiring $\chi^{-1}(t_c) = 0$, and hence from eq.(IV.23), to order of u,

$$t_c = -4u(n+2) \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{t_c + Kk^2} \approx -\frac{4u(n+2)K_d\Lambda^{d-2}}{(d-2)K} < 0. \tag{IV.25}$$

How does the perturbed susceptibility diverge at the shifted critical point? From eq.(IV.23),

$$\begin{aligned}\chi^{-1}(t) - \chi^{-1}(t_c) &= t - t_c + 4u(n+2) \int \frac{d^d\mathbf{k}}{(2\pi)^d} \left(\frac{1}{t + Kk^2} - \frac{1}{t_c + Kk^2} \right) \\ &= (t - t_c) \left[1 - \frac{4u(n+2)}{K^2} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{k^2(k^2 + (t-t_c)/K)} + \mathcal{O}(u^2) \right].\end{aligned} \tag{IV.26}$$

(In going from the first equation to the second, we have changed the position of t_c from one denominator to another. Since $t_c = \mathcal{O}(u)$, the corrections to such change only appear at $\mathcal{O}(u^2)$.) The final integral has dimensions of $\left[k^{d-4}\right]$. For $d > 4$ it is dominated by the largest momenta and scales as $K_d\Lambda^{d-4}$. For $2 < d < 4$, the integral is convergent at both limits. Its magnitude is therefore set by the momentum scale $\xi^{-1} = \sqrt{t - t_c)/K}$, which can be used to make the integrand dimensionless. Hence, in these dimensions,

$$\chi^{-1}(t) = (t - t_c) \left[1 - \frac{4u(n+2)}{K^2} c \left(\frac{K}{t - t_c} \right)^{2-d/2} + \mathcal{O}(u^2) \right], \tag{IV.27}$$

where c is a constant. For $d < 4$, the correction term at the order of u diverges at the phase transition, masking the unperturbed singularity of χ with $\gamma = 1$. Thus the perturbation series is inherently inapplicable for describing the divergence of susceptibility in $d < 4$. The same situation applies in calculating any other quantity perturbatively. Although, we start by treating u as the perturbation parameter, it is important to realize that it is not dimensionless; u/K^2 has dimensions of $(\text{length})^{d-4}$. The perturbation series for any quantity then takes the form $X(t, u) = X_0(t)[1 + f(ua^{4-d}/K^2, u\xi^{4-d}/K^2)]$, where f is a power series. The two length scales a and ξ are available to construct dimensionless variables. Since ξ diverges close to the critical point, there is an inherent failure of the perturbation series. The effective (dimensionless) perturbation parameter diverges at t_c and is not small.