

## III. The Scaling Hypothesis

### III.A The Homogeneity Assumption

In the previous chapters, the singular behavior in the vicinity of a continuous transition was characterized by a set of critical exponents  $\{\alpha, \beta, \gamma, \delta, \nu, \eta, \dots\}$ . The saddle-point estimates of these exponents were found to be unreliable due to the importance of fluctuations. Since the various thermodynamic quantities are related, these exponents can not be independent of each other. The goal of this chapter is to discover the relationships between them, and to find the minimum number of independent exponents needed to describe the critical point.

The non-analytical structure is a coexistence line for  $t < 0$  and  $h = 0$ , that terminates at the critical point  $t = h = 0$ . The various exponents describe the leading singular behavior of a thermodynamic quantity  $Q(t, h)$ , in the vicinity of this point. A basic quantity in the canonical ensemble is the free energy, which in the saddle-point approximation is given by

$$f(t, h) = \min_m \left[ \frac{t}{2} m^2 + u m^4 - h \cdot m \right] \propto \begin{cases} -\frac{t^2}{u} & \text{for } h = 0, t < 0 \\ -\frac{h^{4/3}}{u^{1/3}} & \text{for } h \neq 0, t = 0 \end{cases}. \quad (\text{III.1})$$

The singularities in the free energy can in fact be described by a single *homogeneous* function<sup>†</sup> in  $t$  and  $h$ , as

$$f(t, h) = |t|^2 g_f (h/|t|^\Delta). \quad (\text{III.2})$$

The function  $g_f$  only depends on the combination  $x \equiv h/|t|^\Delta$ , where  $\Delta$  is known as the *gap exponent*. The asymptotic behavior of  $g_f$  is easily obtained by comparing eqs.(III.1) and (III.2). The  $h = 0$  limit is recovered if  $\lim_{x \rightarrow 0} g_f(x) \sim 1/u$ , while to get the proper power of  $h$ , we must set  $\lim_{x \rightarrow \infty} g_f(x) \sim x^{4/3}/u^{1/3}$ . The latter implies  $f \sim |t|^2 h^{4/3}/(u^{1/3}|t|^{4\Delta/3})$ .

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<sup>†</sup> In general, a function  $f(x_1, x_2, \dots)$  is homogeneous if

$$f(b^{p_1} x_1, b^{p_2} x_2, \dots) = b^{p_f} f(x_1, x_2, \dots),$$

for any rescaling factor  $b$ . With the proper choice of  $b$  one of the arguments can be removed, leading to a scaling forms used in this section.

Since  $f$  can have no  $t$  dependence along  $t = 0$ , the gap exponent (corresponding to Eq.(III.1) has the value

$$\Delta = \frac{3}{2} . \quad (\text{III.3})$$

The assumption of homogeneity is that, on going beyond the saddle-point approximation, the singular form of the free energy (and any other thermodynamic quantity) retains the homogeneous form

$$f_{\text{sing.}}(t, h) = |t|^{2-\alpha} g_f (h/|t|^\Delta) . \quad (\text{III.4})$$

The actual exponents  $\alpha$  and  $\Delta$  depend on the critical point being considered. The dependence on  $t$  is chosen to reproduce the heat capacity singularity at  $h = 0$ . The singular part of the energy is obtained from (say for  $t > 0$ )

$$\begin{aligned} E_{\text{sing.}} &\sim \frac{\partial f}{\partial t} \sim (2 - \alpha)t^{1-\alpha} g_f (h/|t|^\Delta) - \Delta h t^{1-\alpha-\Delta} g'_f (h/|t|^\Delta) \\ &\sim t^{1-\alpha} \left[ (2 - \alpha)g_f (h/|t|^\Delta) - \frac{\Delta h}{t^\Delta} g'_f (h/|t|^\Delta) \right] \equiv t^{1-\alpha} g_E (h/|t|^\Delta) . \end{aligned} \quad (\text{III.5})$$

Thus the derivative of one homogeneous function is another. Similarly, the second derivative takes the form (again for  $t > 0$ )

$$C_{\text{sing.}} \sim -\frac{\partial^2 f}{\partial t^2} \sim t^{-\alpha} g_C (h/|t|^\Delta) , \quad (\text{III.6})$$

reproducing the scaling  $C_{\text{sing.}} \sim t^{-\alpha}$ , as  $h \rightarrow 0$ .

It may appear that we have the freedom to postulate a more general form

$$C_{\pm}(t, h) = |t|^{-\alpha_{\pm}} g_{\pm} (h/|t|^{\Delta_{\pm}}) , \quad (\text{III.7})$$

with different functions and exponents for  $t > 0$  and  $t < 0$ , that match at  $t = 0$ . However, this is ruled out by the condition that the free energy is analytic everywhere except on the coexistence line for  $h = 0$  and  $t < 0$ , as proven as follows: Consider a point at  $t = 0$  and finite  $h$ . By assumption, the function  $C$  is perfectly analytic in the vicinity of this point, expandable in a Taylor series,

$$C(t \ll h^\Delta) = \mathcal{A}(h) + t\mathcal{B}(h) + \mathcal{O}(t^2) . \quad (\text{III.8})$$

Furthermore, the same expansion should be obtained from both  $C_+$  and  $C_-$ . But eq.(III.7) leads to the expansions,

$$C_{\pm} = |t|^{-\alpha_{\pm}} \left[ A_{\pm} \left( \frac{h}{|t|^{\Delta_{\pm}}} \right)^{p_{\pm}} + B_{\pm} \left( \frac{h}{|t|^{\Delta_{\pm}}} \right)^{q_{\pm}} + \dots \right] , \quad (\text{III.9})$$

where  $\{p_{\pm}, q_{\pm}\}$  are the leading powers in asymptotic expansions of  $g_{\pm}$  for large arguments, and  $\{A_{\pm}, B_{\pm}\}$  are the corresponding pre-factors. Matching to the Taylor series in eq.(III.8) requires  $p_{\pm}\Delta_{\pm} = -\alpha_{\pm}$  and  $q_{\pm}\Delta_{\pm} = -(1 + \alpha_{\pm})$ , and leads to

$$C_{\pm}(t \ll h^{\Delta}) = A_{\pm}h^{-\alpha_{\pm}/\Delta_{\pm}} + B_{\pm}h^{-(1+\alpha_{\pm})/\Delta_{\pm}}|t| + \dots \quad (\text{III.10})$$

Continuity at  $t = 0$  now forces  $\alpha_{+}/\Delta_{+} = \alpha_{-}/\Delta_{-}$ , and  $(1 + \alpha_{+})/\Delta_{+} = (1 + \alpha_{-})/\Delta_{-}$ , which in turn implies

$$\begin{cases} \alpha_{+} = \alpha_{-} \equiv \alpha \\ \Delta_{+} = \Delta_{-} \equiv \Delta \end{cases}. \quad (\text{III.11})$$

Despite using  $|t|$  in the postulated scaling form, we can still ensure the analyticity of the function at  $t = 0$  for finite  $h$  by appropriate choice of parameters, e.g. by setting  $B_{-} = -B_{+}$  to match Eq.(III.10) to the analytic form in Eq.(III.8). Having established this result, we can be somewhat careless henceforth in replacing  $|t|$  in the scaling equations with  $t$ . Naturally these arguments apply to any quantity  $Q(t, h)$ .

Starting from the free energy in eq.(III.4), we can compute the singular parts of other quantities of interest:

- The *magnetization* is obtained from

$$m(t, h) \sim \frac{\partial f}{\partial h} \sim |t|^{2-\alpha-\Delta} g_m(h/|t|^{\Delta}). \quad (\text{III.12})$$

In the limit  $x \rightarrow 0$ ,  $g_m(x)$  is a constant, and

$$m(t, h = 0) \sim |t|^{2-\alpha-\Delta}, \quad \implies \quad \beta = 2 - \alpha - \Delta. \quad (\text{III.13})$$

On the other hand, if  $x \rightarrow \infty$ ,  $g_m(x) \sim x^p$ , and

$$m(t = 0, h) \sim |t|^{2-\alpha-\Delta} \left( \frac{h}{|t|^{\Delta}} \right)^p. \quad (\text{III.14})$$

Since this limit is independent of  $t$ , we must have  $p\Delta = 2 - \alpha - \Delta$ . Hence

$$m(t, h = 0) \sim h^{(2-\alpha-\Delta)/\Delta}, \quad \implies \quad \delta = \Delta/(2 - \alpha - \Delta) = \Delta/\beta. \quad (\text{III.15})$$

- Similarly, the *susceptibility* is computed as

$$\chi(t, h) \sim \frac{\partial m}{\partial h} \sim |t|^{2-\alpha-2\Delta} g_{\chi}(h/|t|^{\Delta}), \quad \Rightarrow \quad \chi(t, h = 0) \sim |t|^{2-\alpha-2\Delta}, \quad \Rightarrow \quad \gamma = 2\Delta - 2 + \alpha. \quad (\text{III.16})$$

Thus, the consequences of the homogeneity assumption are:

- (1) The singular parts of all critical quantities  $Q(t, h)$ , are homogeneous, with the same exponents above and below the transition.
- (2) Because of the interconnections via thermodynamic derivatives, the same gap exponent  $\Delta$ , occurs for all such quantities.
- (3) All (bulk) critical exponents can be obtained from only *two* independent ones, e.g.  $\alpha$  and  $\Delta$ .
- (4) As a result of the above, there are a number of *exponent identities*. For example, eqs.(III.13), (III.15), and (III.16) imply

$$\alpha + 2\beta + \gamma = \alpha + 2(2 - \alpha - \Delta) + (2\Delta - 2 + \alpha) = 2 \quad (\text{Rushbrooke's Identity}),$$

$$\delta - 1 = \frac{\Delta}{2 - \alpha - \Delta} - 1 = \frac{2\Delta - 2 + \alpha}{2 - \alpha - \Delta} = \frac{\gamma}{\beta} \quad (\text{Widom's Identity}).$$

(III.17)

These identities can be checked against the following table of critical exponents. The first three rows are based on a number of theoretical estimates in  $d = 3$ ; the last row comes from an exact solution in  $d = 2$ . The exponent identities are completely consistent with these values, as well as with all reliable experimental data.

	$\alpha$	$\beta$	$\gamma$	$\delta$	$\nu$	$\eta$
$n = 1$	0.11	0.32	1.24	4.9	0.63	0.04
$n = 2$	-0.01	0.35	1.32	4.7	0.67	0.04
$n = 3$	-0.11	0.36	1.39	4.9	0.70	0.04
$n = 1$	0	1/8	7/4	15	1	1/4

### **III.B Divergence of the Correlation Length**

The homogeneity assumption relates to the free energy and quantities derived from it. It says nothing about the behavior of correlation functions. An important property of a critical point is the divergence of the correlation length, which is responsible for, and can be deduced from, diverging response functions. In order to obtain an identity involving the exponent  $\nu$  for the divergence of the correlation length, we replace the homogeneity assumption for the free energy, with the following *two* conditions:

- (1) The correlation length  $\xi$  has a homogeneous form,

$$\xi(t, h) \sim |t|^{-\nu} g(h/|t|^\Delta). \quad (\text{III.18})$$

(For  $t = 0$ ,  $\xi$  diverges as  $h^{-\nu_h}$  with  $\nu_h = \nu/\Delta$ .)

- (2) Close to criticality, the correlation length  $\xi$  is the most important length in the system, and is *solely* responsible for singular contributions to thermodynamic quantities.

The second condition determines the singular part of the free energy. Since  $\ln Z(t, h)$  is *extensive* and *dimensionless*, it must take the form

$$\ln Z = \left(\frac{L}{\xi}\right)^d \times g_s + \left(\frac{L}{a}\right)^d \times g_a \quad , \quad (\text{III.19})$$

where  $g_s$  and  $g_a$  are non-singular functions of dimensionless parameters ( $a$  is an appropriate microscopic length). The singular part of the free energy comes from the first term, and behaves as

$$f_{\text{sing.}}(t, h) \sim \frac{\ln Z}{L^d} \sim \xi^{-d} \sim |t|^{d\nu} g_f(h/|t|^\Delta) . \quad (\text{III.20})$$

A simple interpretation of the above result is obtained by dividing the system into units of the size of the correlation length. Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as  $(L/\xi)^d$ , leading to eq.(III.19).

The consequences of the above assumptions are:

- (1) The homogeneity of  $f_{\text{sing.}}(t, h)$  emerges naturally.
- (2) We obtain the additional exponent relation

$$2 - \alpha = d\nu \quad (\text{Josephson's identity}) \quad . \quad (\text{III.21})$$

Identities obtained from the generalized homogeneity assumption involve the space dimension  $d$ , and are known as *hyperscaling relations*. The relation between  $\alpha$  and  $\nu$  is consistent with the exponents in the above table. However, it does not agree with the saddle point values,  $\alpha = 0$  and  $\nu = 1/2$ , which are valid for  $d > 4$ . Any theory of critical behavior must thus account for the validity of this relation in low dimension, and its breakdown for  $d > 4$ .

### III.C Critical Correlation Functions and Self-Similarity

One exponent that has not so far been accounted for is  $\eta$ , describing the decay of correlation functions at criticality. Exactly at the critical point, the correlation length is infinite, and there is no other length scale (except sample size) to cut off the decay of correlation functions. Thus all correlations decay as a power of the separation. As discussed in the previous chapter, the magnetization correlations fall off as

$$G_{m,m}^c(\mathbf{x}) \equiv \langle m(\mathbf{x})m(\mathbf{0}) \rangle - \langle m \rangle^2 \sim 1/|\mathbf{x}|^{d-2+\eta}. \quad (\text{III.22})$$

Similarly, we can define an exponent  $\eta'$  for the decay of energy–energy correlations as

$$G_{E,E}^c(\mathbf{x}) = \langle \mathcal{H}(\mathbf{x})\mathcal{H}(\mathbf{0}) \rangle - \langle \mathcal{H} \rangle^2 \sim 1/|\mathbf{x}|^{d-2+\eta'}. \quad (\text{III.23})$$

Away from criticality, the power laws are cut off for distances  $|\mathbf{x}| \gg \xi$ . As the response functions can be obtained from integrating the connected correlation functions, there are additional exponent identities, such as (Fisher’s identity)

$$\chi \sim \int d^d \mathbf{x} G_{mm}^c(\mathbf{x}) \sim \int_0^\xi \frac{d^d x}{|x|^{d-2+\eta}} \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)}, \quad \implies \quad \gamma = (2-\eta)\nu \quad . \quad (\text{III.24})$$

Similarly, for the heat capacity,

$$C \sim \int d^d \mathbf{x} G_{EE}^c(\mathbf{x}) \sim \int_0^\xi \frac{d^d x}{|x|^{d-2+\eta'}} \sim \xi^{2-\eta'} \sim |t|^{-\nu(2-\eta')}, \quad \implies \quad \alpha = (2-\eta')\nu \quad . \quad (\text{III.25})$$

As before, two *independent* exponents are sufficient to describe all singular critical behavior.

An important consequence of these scaling ideas is that the critical system has an additional *dilation symmetry*. Under a change of scale, the critical correlation functions behave as

$$G_{\text{critical}}(\lambda \mathbf{x}) = \lambda^p G_{\text{critical}}(\mathbf{x}). \quad (\text{III.26})$$

This implies a *scale invariance* or *self-similarity*: if a snapshot of the critical system is blown up by a factor of  $\lambda$ , apart from a change of contrast (multiplication by  $\lambda^p$ ), the resulting snapshot is statistically similar to the original one. Such statistical self-similarity is the hallmark of *fractal* geometry. As discussed by Mandelbrot, many naturally occurring forms (clouds, shore-lines, river basins, etc.) exhibit such behavior. The Landau–Ginzburg probability was constructed on the basis of *local* symmetries such as rotation invariance. If we could add to the list of constraints the requirement of *dilation symmetry*, the resulting probability would indeed describe the critical point. Unfortunately, it is not possible to directly see how such a requirement constrains the effective Hamiltonian. One notable exception is in  $d = 2$ , where dilation symmetry implies conformal invariance, and a lot of information can be obtained by constructing conformally invariant theories. We shall instead prescribe a less direct route of following the effects of the dilation operation on the effective energy; the *renormalization group* procedure.