MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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8.422 Atomic Physics II: Lecture 21 – Models of Decoherence

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1 Overview

The optical Bloch equations are an effective tool for studying atoms and light forces, specifically when we are interested in statistical averages, and usually for interactions with classical fields with many photons. There are many other important situations, however, where we wish to study open quantum system dynamics outside of such approximations, such as when a single atom interacts with one or just a few photons. This realm of physics is now becoming experimentally accessible, with cavity QED, ion trap, and optical dipole trap systems, and such experiments motivate the development of more precise theoretical tools for describing open quantum system dynamics, and decoherence, at the level of single quanta.

In this lecture, we focus on models of decoherence. First, we show that the quantum monte carlo method, introduced in the last lecture, gives dynamics identical to the optical Bloch equation. We then define decoherence, and give some perspective on the general study of the decoherence of two-level systems. This study is illustrated specifically by three interesting models for phase damping. Decoherence also occurs when two-level atoms interact with weak light fields, even if the photons are in a coherent state, as shown by the collapses and revivals in the solutions to the Jaynes-Cummings Hamiltonian.

2 Equivalence of QMCWF and OBE

Consider a single two-level atom (levels $|e\rangle$ and $|g\rangle$) initially in the state $|\psi\rangle = a|g\rangle + b|e\rangle$, with the free atom Hamiltonian H_0, interacting with a single mode of the electromagnetic field (initially in the vacuum state $|0\rangle$). The quantum monte-carlo wavefunction (QMCWF) technique can be used to study this scenario, and model spontaneous emission, as embodied in this procedure:

- **1.** Compute $dp = \Gamma dt\langle e|\psi\rangle\langle\psi|e\rangle$
- **2.** Let ϵ be a uniform random number in $[0, 1]$
- **3.** If $\epsilon < dp$ then $|\psi\rangle \leftarrow |g\rangle$
- **4.** If $\epsilon \geq dp$ then $|\psi\rangle \leftarrow \frac{e^{-iHdt}|\psi\rangle}{\sqrt{1-dp}}$
- **5.** Go to **1.**

Here, $H = H_0 - i\frac{\Gamma}{2}|e\rangle\langle e|$ is a non-hermitean Hamiltonian that generates the non-unitary evolution during the differential timestep dt, when no photon is emitted by the atom. Note that $dp = 1 - \langle\psi|e^{iH^\dagger dt}e^{-iHdt}|\psi\rangle$.

The equivalence of QMCWF and the optical Bloch equations (OBE) is shown by demonstrating that the evolution of the density matrix

$$\rho(t) = \overline{|\psi(t)\rangle\langle\psi(t)|} \tag{1}$$

satisfies the OBE, where the average (denoted by the overline) is taken over instances of running the QMCWF procedure. This follows from computing the density matrix for the state after one QMCWF procedure step:

$$\begin{aligned}
\rho(t+dt) &= dp\,|g\rangle\langle g| + (1-dp)\frac{e^{-iHdt}|\psi\rangle\langle\psi|e^{+iH^\dagger dt}}{1-dp} && (2)\\
&\approx dp\,|g\rangle\langle g| + (1-iHdt)\rho(t)(1+iH^\dagger dt) && (3)\\
&\approx dp\,|g\rangle\langle g| + \rho(t) - i(H\rho(t)-\rho H^\dagger)dt && (4)\\
&= \Gamma dt\langle e|\rho(t)|e\rangle\,|g\rangle\langle g| + \rho(t) - i(H\rho(t)-\rho H^\dagger)dt && (5)\\
&= \Gamma dt|g\rangle\langle e|\rho(t)|e\rangle\langle g| + \rho(t) - i[H_0,\rho(t)] - \frac{\Gamma}{2}\left(|e\rangle\langle e|\rho(t)+\rho|e\rangle\langle e|\right). && (6)
\end{aligned}$$

Writing this as a coarse-grained differential equation, taking the limit of small dt, we find

$$\frac{\rho(t+dt)-\rho(t)}{dt} \approx \frac{d}{dt}\rho(t) = -i[H_0,\rho(t)] - \frac{\Gamma}{2}\left(|e\rangle\langle e|\rho(t)+\rho(t)|e\rangle\langle e|\right) + \Gamma dt|g\rangle\langle e|\rho(t)|e\rangle\langle g|\,. \tag{7}$$

This is the OBE.

3 Decoherence: definition and perspective

Spontaneous emission, which is mainly what we have studied so far, with the OBE and QMCWF, is described by the density matrix evolution

$$\rho(t=0) = \begin{bmatrix} 1-c & b^* \\ b & c \end{bmatrix} \longrightarrow \rho(t) = \begin{bmatrix} 1-ce^{-\Gamma t} & b^*e^{-\Gamma t/2} \\ be^{-\Gamma t/2} & ce^{-\Gamma t} \end{bmatrix}. \tag{8}$$

Here, Γ is the spontaneous emission rate of the atom, parameterizing the loss of energy from the atom (the decay of the diagonal elements of ρ to $|g\rangle\langle g|$), and the simultaneous loss of phase coherence of the atom (the decay of the off-diagonal elements of ρ to zero). More generally, however, we can have two parameters which describe separate decay of the diagonal and off-diagonal elements,

$$\rho(t) = \begin{bmatrix} 1-ce^{-t/T_1} & b^*e^{-t/T_2} \\ be^{-t/T_2} & ce^{-t/T_1} \end{bmatrix}, \tag{9}$$

where T_1 is the energy loss rate, and T_2 is the rate of loss of quantum coherence, also known as the phase damping rate. Both of these effects are known as decoherence.

In particular, we define decoherence as *any process which can turn pure states into statistical mixtures* (mixed states). A density matrix ρ is *pure* if either $\mathrm{Tr}(\rho^2)=1$, or $\rho=|\psi\rangle\langle\psi|$ for some state $|\psi\rangle$, or the entropy $S(\rho)=-\mathrm{Tr}(\rho\log\rho)=0$. Otherwise, it is mixed.

Phase damping, which gives rise to T_2, is in a sense the most "quantum" kind of noise; it comes along with spontaneous emission, but can also exist by itself. Historically, the term "decoherence" has sometimes been identified exclusively with phase damping, because of its important role in the emergence of classical behavior from quantum systems, but today, decoherence is used as a much more general term, because there are so many ways to loose coherence from a quantum system other than just phase damping.

4 Phase damping – three models

Let us consider the physical origin of phase damping, because it will teach us an important fact about models of decoherence for single quantum systems. In particular, we consider three different physical models of phase damping, on a two-level atom.

4.1 Random phase noise

A two-level atom of frequency ω_0 excited by far off-resonance light, $\omega \gg \omega_0$, experiences an AC stark shift of amount proportional to the light intensity. If this intensity fluctuates, then the atom's phase is randomly modulated, causing the evolution

$$|e\rangle \to e^{i\theta}|e\rangle\,, \tag{10}$$

where θ is the phase imparted by the AC stark shift. Suppose θ is modeled as a gaussian distributed random variable, with mean zero and variance $2\lambda t$, such that

$$\mathrm{prob}(\theta) = \frac{1}{\sqrt{4\pi\lambda t}} e^{-\theta^2/4\lambda t}\,. \tag{11}$$

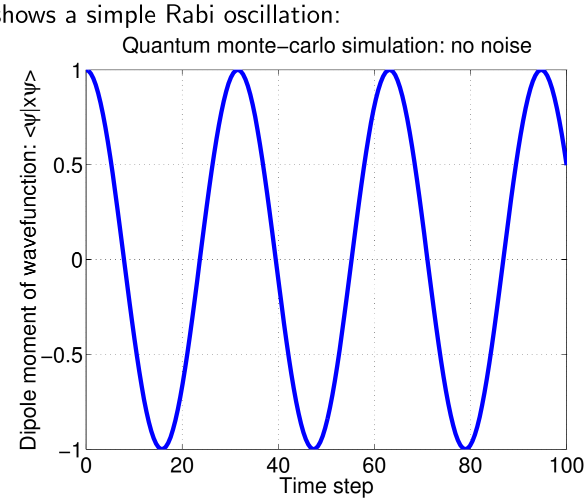
If the initial state of the atom is $\psi\rangle = a|g\rangle + b|e\rangle$, then after time t, it evolves into the average state described by the density matrix

$$\rho(t) = \int_{-\infty}^{+\infty} \begin{bmatrix} |a|^2 & ab^* e^{i\theta} \\ a^* b e^{-i\theta} & |b|^2 \end{bmatrix} \mathrm{prob}(\theta)\, dt\,, \tag{12}$$

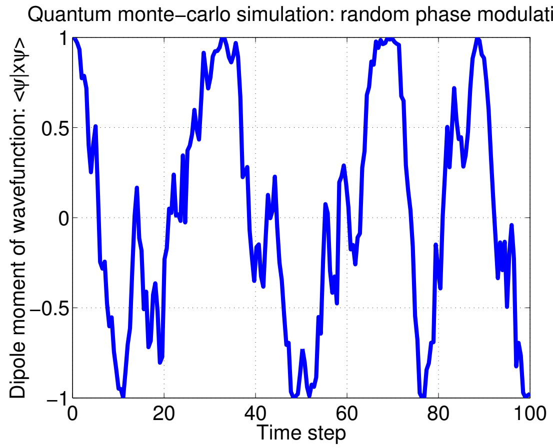
which is found to be

$$\rho(t) = \begin{bmatrix} |a|^2 & ab^* e^{-\lambda t} \\ a^* b e^{-\lambda t} & |b|^2 \end{bmatrix}\,. \tag{13}$$

If the atomic Hamiltonian is $H_0 = \Omega|e\rangle\langle e|$, and the atom's initial state is $|\psi\rangle = (|g\rangle + |e\rangle)/\sqrt{2}$, then the dipole moment of the atom shows a simple Rabi oscillation:



When random phase noise is imposed on the atom, then its dipole moment decays with time. The evolution of a *single* atom, according to the "trajectory" described by the random walk of Eq.(10), is a noisy Rabi oscillation:



4.2 Elastic collisions

Another physical origin for phase damping is elastic collisions. Assume the two-level atom bounces along a waveguide, interacting with the walls without loosing kinetic energy, but changing its trajectory slightly at each bounce, in a manner depending on the state of the atom. This can be modeled by a Hamiltonian interacting the atom with a single mode environment,

$$H_{SE} = |e\rangle\langle e| \otimes \Big[\gamma|0\rangle\langle 1| + \gamma^*|1\rangle\langle 0|\Big] \,, \tag{14}$$

with coupling constant γ. During a small differential timestep dt, an initial atomic state $a|g\rangle + b|e\rangle$ coupled to an environment $|0\rangle$ evolves to become

$$\begin{aligned} (a|g\rangle + b|e\rangle) \otimes |0\rangle &\to a|g\rangle|0\rangle + b|e\rangle(\cos\theta|0\rangle + \sin\theta|1\rangle) & (15)\\ &= \Big[a|g\rangle + b\cos\theta|e\rangle\Big]\,|0\rangle + \Big[b\sin\theta|e\rangle\Big]\,|1\rangle \,, & (16) \end{aligned}$$

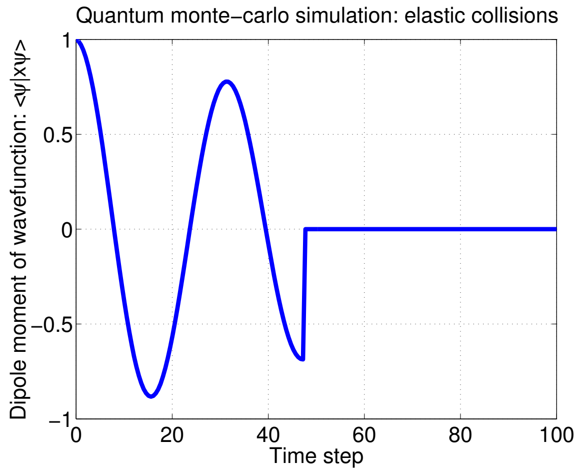
where $e^{-\lambda dt} = \cos\theta$. This expression is very similar to that obtained for the gedankenexperiment used in the QMCWF model of spontaneous emission; the difference is that when a photon is observed in the environment, the atom does not collapse into $|g\rangle$, but rather, into $|e\rangle$. In other words, it does not loose energy; it only looses *information* about what state it was in, prior to the collapse.

Just as in the proof of the equivalence of QMCWF to the OBE, we can compute the density matrix evolution which this model gives rise to, by writing down an expression for $\rho(t)$, based on Eq.(16),

$$\begin{aligned} \rho(t) &= \Big[a|g\rangle + b\cos\theta|e\rangle\Big]\Big[\langle g|a^* + \langle e|b^*\cos\theta\Big] + |b|^2\sin^2\theta|e\rangle\langle e| & (17)\\ &= \begin{bmatrix} |a|^2 & ab^*e^{-\lambda t} \\ a^*be^{-\lambda t} & |b|^2 \end{bmatrix} \,. & (18) \end{aligned}$$

Note that this is exactly the same evolution as we obtained for the random phase noise model, Eq.(13).

Despite the density matrix evolution being identical to that of the random phase model, the elastic collision model implies a different single-particle evolution trajectory. In contrast to the noisy Rabi oscillations previously seen, for the elastic collisions, the atomic state initially decays, then *jumps* into the $|e\rangle$ state at some random time; this is illustrated by this sample trajectory:



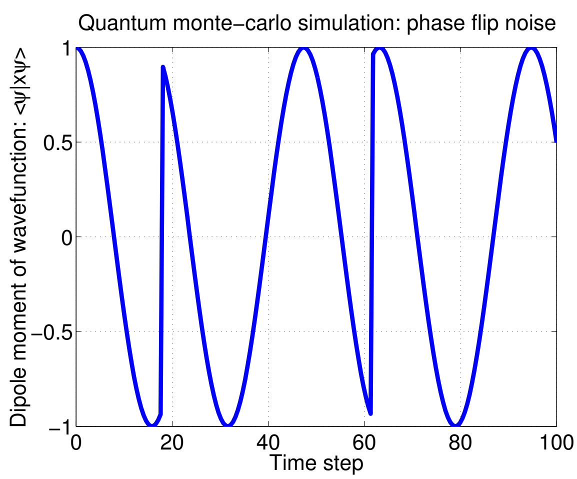
4.3 Phase flips

A third physical model for phase damping is the following (somewhat unphysical!) scenario. Suppose the two-level atom is subject to a force which randomly flips the phase of the atom by -1, changing $|e\rangle \rightarrow -|e\rangle$, with probability $(1-e^{-\lambda t})/2$ at each moment in time. The density matrix for this evolution is thus

$$\rho(t) = \frac{1+e^{-\lambda t}}{2}\Big[a|g\rangle + b|e\rangle\Big]\Big[\langle g|a^* + \langle e|b^*\Big] + \frac{1-e^{-\lambda t}}{2}\Big[a|g\rangle - b|e\rangle\Big]\Big[\langle g|a^* - \langle e|b^*\Big] \tag{19}$$

$$= \left[\begin{array}{cc} |a|^2 & ab^*e^{-\lambda t} \\ a^*be^{-\lambda t} & |b|^2 \end{array}\right]. \tag{20}$$

This is again the same density matrix dynamics as previously obtained for the random phase noise and elastic collision models. However, the trajectories of individual evolutions is different; at each moment in time, the two-level atom either remains *completely unchanged*, or its excited state flips sign, inverting its dipole moment:



4.4 Discussion

We have seen three models of phase damping, all of which produce the same density matrix evolution, but each of which has very different microscopic trajectories for individually evolving systems.

Which is correct?

The answer is that all of them are correct, and yet none are. Any of the three can be used for physical intuition and interpretation, but only as long as the only conclusions drawn depend on statistical averages. In fact, in the absence of control over the enviroment, *no experiment can distinguish between phase damping processes* described by these three models, even in principle. This strong statement arises from the fact that there are an infinite number of ways a (mixed) density matrix can be written as a statistical mixture of pure states; correspondingly, there are an infinite number of "unravelings" of density matrix time evolutions, into statistical evolutions of pure state wavefunctions.

The freedom of *interpretation* which arises in studying decoherence processes arises from a unitary degree of freedom. For example, in the QMCWF model, a gedankenexperiment is performed, in which the atom is allowed to decay and the emitted photon is captured. The evolution of the atom cannot depend on what measurement basis is used for the photodetection. This basis choice is a unitary transform which can be chosen arbitrarily in the gedankenexperiment, and different choices lead to the different unravelings of the master equation into trajectories.

5 Jaynes-Cummings collapse and revival

The models of decoherence we have considered so far involve interaction between system and environment, where each part is explicitly identified. There are more subtle cases, however, where decoherence arises even when no explicit separation is apparent at first sight. One important example of this is the interaction of two-level atoms with coherent states of light of low photon number.

In our derivation of the optical Bloch equations, following API, we noted that when the electromagnetic field is a coherent state, the atom-light interaction can be split into two pieces: a classical term $\mathcal{E}_0 \cos \omega t$ driving Rabi oscillations of the atom, and a quantum term involving $E_\perp(0)$, the vacuum state. At this point, an important approximation was made, that contributions from these two sources could be studied independently, such that Rabi oscillations and spontaneous emission happened separately. This approximation is invalid when the mean photon number $\bar{n}$ is small, and when the photons interact coherently with an atom for a long time, as for example with atoms in an optical cavity. Indeed, for small $\bar{n}$, the driven Rabi oscillations *decay*, in a manner reminiscent of decoherence.

This behavior can be quantitatively derived by studying the explicit solutions to the Jaynes-Cummings Hamiltonian. Recall this to be

$$H_{JC} = \frac{\hbar\Omega}{2} \left(a^\dagger |g\rangle\langle e| + a|e\rangle\langle g| \right) , \tag{21}$$

with eigenstates

$$|\pm, n\rangle = \frac{1}{\sqrt{2}} \left[|e, n\rangle \pm |g, n+1\rangle \right] \tag{22}$$

and eigenvalues

$$E_{\pm,n} = \hbar(n+1)\omega \pm \frac{\hbar\Omega}{2}\sqrt{n+1} \,. \tag{23}$$

Consider a single atom interacting with n photons; this is the state

$$|\psi(0)\rangle = |e, n\rangle = \frac{1}{\sqrt{2}} \left[|+, n\rangle + |-, n\rangle \right] \,. \tag{24}$$

Time-evolution following H_{JC} gives

$$|\psi(t)\rangle = \frac{e^{-i(n+1)\omega t}}{\sqrt{2}} \left[e^{-i\Omega\sqrt{n+1}t/2}|+,n\rangle + e^{+i\Omega\sqrt{n+1}t/2}|-,n\rangle \right] . \tag{25}$$

The atom is thus found to be in the excited state with probability

$$P_e(t) = |\langle e,n|\psi(t)\rangle|^2 = \cos^2\left(\frac{\Omega\sqrt{n+1}}{2}t\right) ; \tag{26}$$

these are the expected driven Rabi oscillations. When the field is not a simple Fock state $|n\rangle$, but rather, a coherent state $|\alpha\rangle$, with average photon number $\bar{n} = |\alpha|^2$, the excited state probability becomes

$$P_e(t) = \sum_n |C_n|^2 |\langle e,n|\psi(t)\rangle|^2 = e^{-\bar{n}} \sum_n \frac{\bar{n}^n}{n!} \frac{1+\cos^2\left(\frac{\Omega\sqrt{n+1}}{2}t\right)}{2} , \tag{27}$$

where C_n is the Poisson distribution of amplitudes for n in the coherent state. For large $\bar{n}$, Rabi oscillations emerge, with Rabi frequency $\Omega\sqrt{\bar{n}}$.

However, for small $\bar{n}$, oscillations of different phases interfere with each other; this leads to complete destruction if the amplitudes are π out of phase with each other after after one period of the Rabi oscillation. This time $T_{\rm collapse}$ is determined by

$$\left(\frac{d}{dn}\sqrt{n+1}\bigg|_{n=\bar{n}} \right) \Omega\sqrt{\bar{n}}\, T_{collapse} = \pi \tag{28}$$

from which the collapse time is found to be

$$T_{\rm collapse} = \frac{2\pi}{\Omega} , \tag{29}$$

where Ω is the vacuum Rabi frequency.

This decay in Rabi oscillations is, however, not a simple decoherence; the exact solution shows a small revival of oscillations after $T_{\rm collapse}$, after which further swings between collapse and revival recur. Unlike the system-environment models of decoherence which underlie the optical Bloch equations, and other master equation models, decoherence here arises due to spread of information from the two-level system into to the infinite number of states in the harmonic oscillator to which the system is coupled.