Lecture 7: The Scaling Theory of Conductance - Part II

Last time, we studied localization and conductance in one dimension using arguments about the scaling behavior of systems with random potentials. In this lecture, we will extend those arguments to higher dimensions to learn about conductance in two and three dimensional systems.

In the last lecture, we first studied some basic properties of the energy levels of independent electrons in a box with a random potential. The energy levels of such a system occur at random energies, with a typical spacing between levels Δ. We then investigated the perturbing effect of the coupling between many such boxes in contact with one another. The typical energy range over which the levels would be shifted by this coupling is of the order of the Thouless energy E_T. Extended (conducting) states of the joint system are realized when E_T is much greater than the typical level spaceing Δ. This is the strong-coupling limit.

The Thouless energy was also related to the dephasing time τ_ϕ through the relation

$$E_T = \frac{\hbar}{\tau_\phi} \tag{7.1}$$

Within our model of microscopic diffusion, we also get a dephasing length L_ϕ

$$L_\phi = \sqrt{D\tau_\phi} \tag{7.2}$$

where D is the microscopic diffusion constant.

Although we live in a three dimensional world, we can realize the physics of lower dimensional systems from bulk systems by keeping some dimensions of the system on the order of the dephasing length. For instance, a long wire with a diameter less than L_ϕ yields a good example of a one dimensional system. A two dimensional system can be realized by evaporating thin films to a thickness of order L_ϕ or less.

We also defined a *localization length* ξ as the length at which the Ohm's law 1/L conductance becomes of order 1. Ordinarily, there are two length scales in a system – the mean free path $\ell = v_F\tau_{el}$, and the localization length ξ. In a strictly one dimensional system, however, all of the back scattering adds up to give 100% reflection after some distance. As a result, there is always localization in a one dimensional system, and $\xi = \ell$.

7.1 Scaling Theory of Conductance: General Formulation

Our idea is to start on a small scale over which we are able to understand the behavior of the system, and then to gradually build up our knowledge at larger distances from this small length-scale knowledge. This approach is very much related to the renormalization group procedure used extensively in statistical physics.

Our main assumption is that the dimensionless conductance g is a function of length scale only

$$g = g(L) \tag{7.3}$$

That is, we assume that when several copies of a small system are stacked together to make up a larger system, the conductance of the larger system depends only on the conductance of the original small system.

We wish to answer the question of how g depends on L. Since the intuitive picture of our scaling procedure involves doubling the system size over and over again, it makes sense to think in a logarithmic scale. Thus we will consider the logarithmic derivative

$$\frac{d\ln g}{d\ln L} = \beta\left[g(L)\right] \tag{7.4}$$

where β is some function of its argument.

Notice that the right hand side of equation (7.4) *depends on L only through the function $g(L)$*. This is exactly what we mean by our assumption of scaling behavior. To determine the form of this function $\beta[g]$, we will examine its behavior for large and small values of the conductance g.

The case of $g \gg 1$ corresponds to the metallic limit, in which we expect the familiar Ohm's Law scaling to hold

$$g \approx \sigma_0 L^{d-2} + \cdots \tag{7.5}$$

where L^{d-2} comes from the ratio of the cross sectional area of the sample to its length.

In this large g limit,

$$\lim_{g\to\infty} \beta[g] = (d-2) \tag{7.6}$$

To modify this result with higher order corrections, we can assume that there exists an analytical expansion for β in powers of $1/g$. Since we are working in the $g \to \infty$ limit, $1/g$ is a good small parameter. To first order in $1/g$, we get

$$\beta[g] \to (d-2) - c\left(\frac{1}{g}\right) \tag{7.7}$$

where c is some unknown constant.

In the opposite limit, we have $g \ll 1$, which corresponds to the case of localization. Here, we expect the conductance to drop off exponentially with length

$$g \approx e^{-L/\xi} \tag{7.8}$$

Differentiating this expression, we get

$$\frac{d\ln g}{d\ln L} = -\frac{1}{\xi}\frac{dL}{d\ln L} \tag{7.9}$$

$$= -\frac{L}{\xi} \tag{7.10}$$

$$= \ln g \tag{7.11}$$

which gives

$$\lim_{g\to 0} \beta[g] = \ln g \tag{7.12}$$

7.2 Scaling Theory of Conductance: Results in 1, 2, and 3 Dimensions

Based on the limiting behavior calculated above, we can draw a qualitative picture of how we expect β to behave in the β-g plane. We analyze the behavior of β in one, two, and three dimensions. In all cases, the conductance *increases* with increasing values of L if $\beta > 0$, and *decreases* with increasing L if $\beta < 0$.

One Dimension: In one dimension, β reaches an asymptotic value of -1 for large values of g. Thus the "flow" of the transformation as L increases is towards *smaller values of g.* Thus as the length of a one dimensional system is increased, the conductance decreases towards 0 and eventually crashes exponentially as g becomes less than unity.

Two Dimensions: For a two dimensional system, the asymptotic limit of β for large values of g is 0. This is an interesting case, as it implies that the conductance of a two dimensional system is *independent* of the system size for a large enough length scale. However, the presence of the correction term in $(1/g)$ indicates that (for $c > 0$) there is still a very slight negative slope to the g versus L curve, meaning that as the system is scaled up the conductance will eventually be sucked in towards $g = 0$ and a localized state will be realized.

To investigate this behavior further, we can integrate $\beta[g]$ to get g as a function of L. For large g and $d = 2$,

$$\frac{d\ln g}{d\ln L} = -\frac{c}{g} \tag{7.13}$$

$$\frac{dg}{d\ln L} = -c \tag{7.14}$$

$$g = g_0 - c\ln\left(\frac{L}{L_0}\right) \tag{7.15}$$

where we have used the initial conditions $L_0 = \ell$, the mean free path characterizing the small length scale over which we began the scaling procedure, and $g_0 = \sigma_{Boltz}$ is the Boltzmann conductivity on this small length scale. Thus we see that the conductance in two dimensions decreases *logarithmically* as L increases.

How do we expect these results to be modified at finite temperatures? Due to fluctuation effects such as the electron-phonon interaction etc, we expect the dephasing time to have power-law scaling with temperature

$$\frac{1}{\tau_\phi} \propto T^P \tag{7.16}$$

Based on this dephasing time, we can define a dephasing length

$$L_\phi = \sqrt{D_0\tau_\phi} \tag{7.17}$$

$$\approx T^{-P/2} \tag{7.18}$$

For length scales greater than this dephasing length, we expect the system to display classical behavior. Thus L_ϕ injects a long distance cut-off, above which g should obey classical Ohm's Law scaling with

$$g = g_0 + \frac{cP}{2}\ln(T\tau_{el}) \tag{7.19}$$

This is the limiting case for perturbative *weak localization* – it is only valid if the correction term is small compared with g_0.

Three Dimensions: In three dimensions, the situation is a little more interesting. The large g limit has $\beta[g]$ positive ($(d-2)=1$), while the small g limit crashes to $-\infty$ like $\ln g$. By continuity, this means that the function $\beta[g]$ must cross the $\beta = 0$ at some point g_c on the g-axis. This point is of considerable interest, as it represents a stationary point of the scaling. If a three dimensional system is realized with a dimensionless conductance equal to the critical conductance g_c, then the conductance will be independent of scale as the system size is changed. However, if the initial conductance is slightly greater than g_c, the flow will take the system to higher and higher values of conductance. On the other hand, if the initial conductance is *less* than g_c, then the conductance will crash to 0 as the system scale is increased. To achieve such conditions in three dimensions, it is necessary to start with an extremely disordered system.

7.3 The Unstable Fixed Point in Three Dimensions

Recall that g_c is the value of the conductance at which $\beta[g_c] = 0$. If the conditions are arranged such that a system in three dimensions has conductance g_c, then the system will exhibit the curious behavior of scale invariance. However, we already showed that the flow for increasing L is *away* from this fixed point for deviations to either side of g_c. Thus the fixed point at g_c is an *unstable* fixed point.

With such interesting mathematical behavior in the vicinity of g_c, you can be sure that there is some interesting physics involved. In fact, this point describes a metal-insulator transition: finite scale systems prepared with conductance less than g_c become asymptotically insulating as $L \to \infty$, while systems with conductance just above g_c become good conductors as $L \to \infty$.

To better understand the behavior close to this transition, we *linearize* the flow around g_c:

$$\beta_L = s\left(\frac{g - g_c}{g_c}\right) \tag{7.20}$$

where β_L is the linearized version of the scaling function β and s is the "slope" of the flow in the β-g plane at $g = g_c$.

Close to the transition (i.e. $\left|\frac{g-g_c}{g_c}\right| \ll 1$), where we expect the linearization of β to be valid, we can replace β in equation (7.4) with β_L

$$\begin{aligned} \frac{d\ln g}{d\ln L} &= \beta_L[g] \\ &= s\left(\frac{g-g_c}{g_c}\right) \end{aligned} \tag{7.21}$$

Using the fact that

$$\begin{aligned} \ln(g/g_c) &= \ln\left(1 + \frac{g-g_c}{g_c}\right) \\ &\approx \frac{g-g_c}{g_c} \end{aligned} \tag{7.22}$$

we can easily solve for the behavior of $g(L)$ in the vicinity of the transition:

$$\frac{d\left(\frac{g-g_c}{g_c}\right)}{d\ln L} = s\left(\frac{g-g_c}{g_c}\right) \tag{7.23}$$

$$\ln \frac{g - g_c}{g_c} = s \ln L \tag{7.24}$$

$$\frac{g(L) - g_c}{g_c} = \left(\frac{L}{L_0}\right)^s \tag{7.25}$$

Thus the conductance diverges from the fixed point as a *power law* in the length scale L of the system. Using our condition for the validity of the power-law scaling region, we can define the localization length ξ through the relation

$$\left|\frac{g(\xi) - g_c}{g_c}\right| = 1 \tag{7.26}$$

Why is it reasonable to associate ξ with L? In the scaling region, the correlation length of the system diverges (as with crucial opalescence) and it is difficult to determine if one is in a metallic or insulating phase. Typical localized states have wave functions that decay exponentially over a length ξ. If we are looking at a length scale L shorter than ξ, it is very difficult to tell the difference between a localized and an extended wave function. However, when L exceeds ξ on the insulating side, the wave function becomes clearly localized and the conductance crashes exponentially to zero. Thus near the transition we expect

$$\left(\frac{\xi}{L_0}\right) = \left|\frac{g - g_c}{g_c}\right|^{-1/s} \tag{7.27}$$

$$= \left|\frac{g - g_c}{g_c}\right|^{-\nu} \tag{7.28}$$

where $\nu = 1/s$ is the critical exponent associated with the correlation length in they study of critical phenomena. Numerical experiments indicate that ν is of order 1.

7.4 Experimentally Observable Consequences

Suppose we have a system on the metallic side of the critical conductance $g > g_c$. For $L \gg \xi$ we expect Ohm's Law type scaling behavior. Thus we can define a large length-scale limiting conductivity $\sigma(L \to \infty)$

$$\sigma(L \to \infty) \approx \sigma(L = \xi) = \frac{\tilde{g}}{\xi^{d-2}} \tag{7.29}$$

This definition makes sense because σ is essentially constant for length scales beyond the localization length ξ. Substituting in equation (7.28) for ξ, we get

$$\sigma(L \to \infty) \propto \left(\frac{g - g_c}{g_c}\right)^{(d-2)\nu} \tag{7.30}$$

7.5 The Model of Microscopic Diffusion

So far, all we have done amounts to little more than fancy guess-work. Although we were able come a very long way using these simple arguments, we still don't have much idea about the values of the coefficients appearing in our equations. To get a handle on this aspect of the theory, we need to consider a microscopic model of our system.

The physical picture we employ is that of the Feynman/Dirac path integral formulation of quantum mechanics. Recall that according to this formulation, the quantum mechanical amplitude for a particle to go from $(\vec{x}_1, t_1)$ to $(\vec{x}_2, t_2)$ is given by

$$\langle \vec{x}_2, t_2 \,|\, \vec{x}_1, t_1 \rangle = \sum_{\text{all paths}} e^{iS/\hbar} \tag{7.31}$$

where S is the *classical* action associated with a particle traversing a particular path from $(\vec{x}_1, t_1)$ to $(\vec{x}_2, t_2)$. Using the method of stationary phase, it can be shown that the primary contribution to this amplitude comes from paths very close to the path satisfying the classical Euler-Lagrange equations. For a free particle, this means that the amplitude is dominated by paths confined to a tube of width proportional to k^{-1} where k is the wave vector of a plane wave connecting $(\vec{x}_1, t_1)$ and $(\vec{x}_2, t_2)$.

In a disordered system, however, the path from $(\vec{x}_1, t_1)$ to $(\vec{x}_2, t_2)$ is better described by a random walk (diffusion). If we associate an amplitude $A_i = |A_i|e^{iS_i/\hbar}$ with each possible random walk path, then the total amplitude A is given by

$$A = \sum_i A_i \tag{7.32}$$

The probability for getting from $(\vec{x}_1, t_1)$ to $(\vec{x}_2, t_2)$ is given by the square of the amplitude $|A|^2$. In general, we expect the phases of different paths to vary considerably, leading to cancellation of the cross-terms in the double-sum for $|A|^2$. Thus we might conclude

$$|A|^2 \approx \sum_i |A_i|^2 \tag{7.33}$$

7.6 Time Reversal Symmetry and the Probability of Return

However, if the system's Hamiltonian exhibits *time reversal symmetry*, we must take more care in calculating the *probability of return*. Consider two paths A_i and $\tilde{A}_i$ that are time-reversed versions of each other. Since the time-reversed amplitude is the same as forward-time version,

$$|A_i + \tilde{A}_i|^2 = 4|A_i|^2 \tag{7.34}$$

which should be compared with the result $2|A_i|^2$ that we got previously assuming cancellation of all cross terms. This shows that quantum interference results in an *enhanced* probability of a particle coming back to where it started from in a system with time-reversal symmetry.

When this result is taken into account in order consideration of propagation through a disordered medium, this can be interpreted as an increase in back scattering that leads to increased resistivity. This phenomenon is actually a result of wave mechanics, and is not restricted to quantum systems. It was first discovered in the 1950s by the study of radar transmission through dense fog. Instead of isotropic scattering, there is an enhancement in the probability of back-scattering (180^o reflections).

What is the classical probability of return? One way to get the answer is to simply solve the diffusion equation:

$$P(r,t) \propto \frac{1}{(Dt)^{d/2}} e^{-r^2/(Dt)} \tag{7.35}$$

The probability of the particle being back where it started at time t is found by setting $r = 0$ in this expression, giving

$$P_{ret} \propto \frac{1}{(Dt)^{d/2}} \tag{7.36}$$

If we wait a long time (on the order of the dephasing time τ_ϕ, then the total probability of coming back through the initial point is given by

$$P_0 = \int_\tau^{\tau_\phi} \frac{dt}{(Dt)^{d/2}} = \begin{cases} \frac{1}{D} \ln\left(\frac{\tau_\phi}{\tau}\right) & d = 2 \\ \frac{1}{D^{3/2}} \left(\frac{\tau_\phi}{\tau}\right)^{-1/2} & d = 3 \end{cases} \tag{7.37}$$

According to equation (7.34), paths which close on themselves acquire an enhanced probability. Thus one expects a reduction of conductivity proportional to P_0. In a previous lecture, we derived the Einstein relation for the conductivity, which yields $g \propto D$. This suggests that in two dimensions

$$\sigma = \sigma_0(1 - \frac{c}{g_0} \ln \frac{\tau_\phi}{\tau}) \tag{7.38}$$

$$g = g_0(1 - \frac{c}{g_0} \ln \frac{\tau_\phi}{\tau}) \tag{7.39}$$

$$g = g_0 - c \ln \frac{\tau}{\tau_\phi} \tag{7.40}$$

We have just seen that time reversal symmetry leads to an enhanced probability of return. If this is correct, then by finding a way to break time reversal symmetry we should see the effect disappear. The easiest way to do this is through the addition of a magnetic field (vector potential). In a magnetic field,

$$A_i \propto e^{iS_i} \rightarrow e^{iS_i + i\oint \vec{A}\cdot d\vec{\ell}} \tag{7.41}$$

$$\tilde{A}_i \propto e^{iS_i} \rightarrow e^{iS_i - i\oint \vec{A}\cdot d\vec{\ell}} \tag{7.42}$$

This time, the cross terms give

$$|A|^2 e^{2i\oint \vec{A}\cdot d\vec{\ell}} = |A|^2 e^{2i\Phi_B/\phi_0} \tag{7.43}$$

where $\phi_0 = hc/e$ is the flux quantum.

The typical magnetic flux through a random-walk return-path is

$$\Phi_B \approx L_\phi^2 B \tag{7.44}$$

When $\Phi_B \gg hc/2e$, the cross terms show up with random phases and the interference is destroyed. In this case, everything goes back to the normal case without the enhanced return probability. From this observation, we can define a magnetic length scale

$$L_B = \sqrt{\frac{hc}{2e}\frac{1}{B}} \tag{7.45}$$

When $L_B < L_\phi$, we should replace L_ϕ with L_B in the correction to the conductivity

$$-c\ln\left(\frac{L_\phi^2}{D\tau}\right) \xrightarrow{B} -c\ln\left(\frac{L_B^2}{D\tau}\right) \tag{7.46}$$

$$\approx \; c\ln B \tag{7.47}$$

To see the effect, we can go to low temperatures and measure the resistivity R as a function of B. Note that the conductivity is predicted to *increase* with magnetic field. This is called negative magnetoresistance and cannot be explained by Boltzmann transport. This phenomenon has been seen experimentally. However, it turns out that there is another logarithmic term in the conductivity for interacting electrons in the presence of disorder which coexists with the weak localization effect. This latter effect has a different magnetic field dependence, which makes magnetoresistance measurements a powerful tool to disentangle the two effects.