

# Lecture 3: Properties of the Response Function

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In this lecture we will discuss some general properties of the response functions  $\chi$ , and some useful relations that they satisfy.

## 3.1 General Properties of $\chi(\vec{q}, \omega)$

Recall that

$$\chi(\vec{q}, \omega) = \frac{\delta n(\vec{q}, \omega)}{U(\vec{q}, \omega)} \quad (3.1)$$

with  $n(\vec{r}, t) \in \mathbb{R}$  and  $U(\vec{r}, t) \in \mathbb{R}$ . Under Fourier transform, this implies

$$n(-\vec{q}, -\omega) = n^*(\vec{q}, \omega) \quad (3.2)$$

$$U(-\vec{q}, -\omega) = U^*(\vec{q}, \omega) \quad (3.3)$$

As a result,

$$\chi(\vec{q}, \omega) = \chi^*(-\vec{q}, -\omega) \quad (3.4)$$

$$\chi''(-\vec{q}, -\omega) = -\chi''(\vec{q}, \omega) \quad (3.5)$$

where  $\chi''$  is the imaginary part of the response function  $\chi(\vec{q}, \omega)$ .

Consider the extension of  $\omega$  to the complex plane. We can then rewrite the expression for  $\chi(\vec{q}, \omega)$  as

$$\chi(\vec{q}, \omega) = \lim_{\eta \rightarrow 0^+} - \sum_n |\langle n | \hat{\rho}_{\vec{q}}^\dagger | \phi_0 \rangle|^2 \left\{ \frac{1}{\omega - (E_n - E_0) + i\eta} - \frac{1}{\omega + (E_n - E_0) + i\eta} \right\} \quad (3.6)$$

Without the  $+i\eta$  term in the energy denominator, there would be singularities (poles) on the real axis whenever  $\omega$  is equal to the spacing between the ground state and some excited state. The presence of  $+i\eta$  pushes these poles just into the lower  $1/2$ -plane, ensuring that  $\chi(\vec{q}, \omega)$  is analytic in the entire upper- $1/2$   $\omega$ -plane including the real axis.

Analyticity of  $\chi(\vec{q}, \omega)$  in the upper- $1/2$  plane is needed to build *causality* into the theory. Consider the response function in time,  $\chi(t)$ :

$$\chi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \chi(\omega) e^{-i\omega t} \quad (3.7)$$

To evaluate this integral, we perform a contour integral in the complex  $\omega$  plane. For  $t < 0$ , closing the contour in the *upper*  $1/2$  plane ensures that  $|e^{-i\omega t}| \rightarrow 0$  on the curved portion of the

contour. Since we have ensured that  $\chi(\omega)$  is analytic in the upper half plane, Cauchy's residue theorem guarantees that the integral over the entire contour is 0. As a result, the piece we need, i.e. the integral from  $-\infty$  to  $\infty$  along the real axis, must also be 0. Thus  $\chi(t) = 0$  for  $t < 0$ , which means that the system cannot respond to a perturbation until *after* the perturbation has occurred.

What about the  $t > 0$  case? In this case, the contour must be closed in the lower  $1/2$  plane to prevent the exponential from blowing up. However, the  $i\eta$  in the energy denominator has pushed the singularities into this region of the complex  $\omega$  plane. Thus the value of the contour integral will be nonzero, and the system will respond to perturbations for  $t > 0$ .

### 3.2 Kramers-Kronig

Consider the integral

$$\oint d\omega' \frac{\chi(\vec{q}, \omega')}{\omega - \omega'} = 0 \quad (3.8)$$

As we have just shown in the previous section, our definition of  $\chi(\vec{q}, \omega)$  ensures that  $\chi(\vec{q}, \omega)$  is analytic in the upper- $1/2$  complex  $\omega$  plane. Although the integrand here has a pole on the real axis due to the  $\omega - \omega'$  in the denominator, by making a hump over this pole we can ensure that the value of the contour integral itself vanishes by Cauchy's residue theorem.

Assuming that  $\chi(\vec{q}, \omega) \rightarrow 0$  as  $|\omega| \rightarrow \infty$ ,

$$0 = \int_{-\infty}^{\infty} d\omega' \chi(\vec{q}, \omega') \text{Pr} \left[ \frac{1}{\omega - \omega'} \right] + i\pi \chi(\vec{q}, \omega) \quad (3.9)$$

where the additional term  $i\pi\chi(\vec{q}, \omega)$  is one half of the contribution from the pole at  $\omega' = \omega$  that we picked up by making a hump over the pole. Thus for fixed  $\vec{q}$ ,

$$\chi'(\vec{q}, \omega) = -\frac{1}{\pi} \text{Pr} \left[ \int_{-\infty}^{\infty} d\omega' \frac{\chi''(\vec{q}, \omega')}{\omega - \omega'} \right] \quad (3.10)$$

$$\chi''(\vec{q}, \omega) = -\frac{1}{\pi} \text{Pr} \left[ \int_{-\infty}^{\infty} d\omega' \frac{\chi'(\vec{q}, \omega')}{\omega - \omega'} \right] \quad (3.11)$$

or equivalently

$$\chi(\vec{q}, \omega) = \lim_{\eta \rightarrow 0^+} - \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\chi(\vec{q}, \omega')}{\omega - \omega' + i\eta} \quad (3.12)$$

The important message from all of this is that the entire response function  $\chi(\vec{q}, \omega)$  can be reconstructed from its imaginary (or real) part *alone*.