MASSACHUSETTS INSTITUTE OF TECHNOLOGY

15.053 – Introduction to Optimization (Spring 2005)

Problem Set 5, Due **at the beginning of class** March 17th, 2005

You will need 49.5 points out of 58 to receive a grade of 2.5.

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1. Prisoner's Dilemma – Play a Game! (6 points)
The Prisoner's Dilemma is a famous problem in game theory. The following website describes the problem, and allows you to simulate several rounds of the game.

http://www.miskatonic.org/pd.html

a) Read the website and learn about the Prisoner's Dilemma. At the bottom of the page, play 10 rounds of the game. Print out your output at the end of the round and turn it in. (If you are working with a partner, then you should both do this and turn in two separate copies.) Who got off with less jail time – you or the computer? What strategy was the computer using? (Note that the computer is programmed to choose from four different strategies, some of which are meaner than others! For fun, you can try playing multiple times, and observe how your performance varies with the strategy the computer chooses.)
b) Do you think the strategy the computer used against you was fair? What choice would you make, if you had to choose whether to cooperate or reject in real life? (There is no wrong answer here.)

2. Game Theory and Linear Programming (10 points)
Consider the following Payoff Matrix, which is similar to the one given in class, except that it has an additional column. The numbers represent payoffs to the row player R.

-2	1	1	3
2	-1	0	-2
1	0	-1	-1

a) Formulate a linear program that will find an optimal mixed strategy for the row player R.
b) Formulate a linear program that will find an optimal mixed strategy for the column player C.
c) Modify the Excel spreadsheet PS5_Problem1.xls, so that it can be used to find the optimal solution for the linear programs in part a) and part b). What is the payoff to the row player if the row player chooses the optimal strategy and the column player decides (unwisely) to choose column 4?

3. Graphical Solution to Game Theory Problems (8 points)
Consider the following payoff matrix for a 2-person 0 sum game.

4	-2
-1	2

a) Use the graphical method to determine the optimal mixed strategy and optimal payoff for the row player.

b) Use the graphical method to determine the optimal mixed strategy and optimal payoff for the column player.

4. Modeling and Game Theory (8 points)
Suppose you and your friend Mike are playing the following (silly) game. You write down an integer between 1 and 20 on a slip of paper. Without showing this slip of paper to Mike, you tell him what it is you have written. You may either lie or tell the truth. Mike must then guess whether or not you have told the truth.

If caught in a lie, you must pay Mike $10; if you are falsely accused of lying, Mike must pay you $5. If you tell the truth and Mike guesses that you have told the truth, then you must pay Mike $1. If you lie and Mike does not guess that you have lied, then you win $5 from Mike.

a) Write down a matrix describing this game. You should be the row player, and Mike should be the column player.
b) Determine your optimal mixed strategy, either using the linear programming technique of problem 1 or the graphical technique of problem 2. What is your optimal payoff under this strategy?

5. Poker and Game Theory (16 points)
This problem is a VERY simplified version of poker, which we call S-poker. It illustrates the value of bluffing in Poker, as well as illustrates how uncertainty about outcomes can be used in game theory. You do not need to know anything about poker for this problem.

S-poker is played by two players, whom we shall call player A and player B. The game is played a number of rounds. At every round, each player receives one card. Player A always receives the 5 of hearts, which is visible to both players. Player B receives either the 4 of hearts or the 6 of hearts. This card is placed down. Player B can see the card, but player A cannot. The probability of player B receiving the 4 of hearts is 3/5, and the probability of receiving the six of hearts is 2/5. This probability is known to both players.

The initial bet per round is $1.

After receiving the card, player B has the option of *doubling* the bet to $2. If player B does not double the bet, then the round is won by the player with the higher card. That is, if B has the 6 of hearts, then B wins a dollar and A loses a dollar. If B has the 4 of hearts, then B loses a dollar and A wins a dollar.

If player B doubles the bet, then player A can either *accept* the double or *reject* the double. If A rejects the double, then A loses $1 and B wins $1, regardless of the card that B has. If player A accepts the double, then the bet becomes $2 and the round is won by the player with the higher card. That is, if B has the 6 of hearts, then B wins $2 and A loses $2. If B has the 4 of hearts, then B loses $2 and player A wins $2.

This describes the entire game.

In parts a)-d), we determine the lowest that player B can guarantee in terms of payoff to player A. We assume that player B always doubles when she has the 6 of hearts. So, a random strategy for B consists of specifying the probability p that B will double when she has the four of hearts. Player A always wins if there is no double. If there is a double, player A needs to respond by either accepting the double or declining the double.

a) Suppose $p = 0$. What is the payoff to player A if A accepts a double? What is the payoff to player A if A rejects a double? What is the expected payoff of the game for player A (assuming that A chooses the better of the two strategies)?
b) Suppose $p = 1$. What is the expected payoff to player A if A accepts a double? What is the payoff to player A if A rejects a double? What is the expected payoff of the game for player A (assuming that A chooses the better of the two strategies)?
c) Suppose $p = .1$. What is the expected payoff to player A if A accepts a double? What is the expected payoff to player A if A rejects a double? What is the expected payoff of the game for player A (assuming that A chooses the better of the two strategies)?
d) Graph the expected payoff of the game for player A as p goes from $p = 0$ to $p = 1$, assuming that player A chooses the better of doubling or not doubling. As in the two variable examples in class, there should be two lines graphed, one according to whether A accepts all doubles and one according to whether A declines all doubles. What is the threshold value of p such that A is indifferent between accepting and declining doubles?

(Note: by choosing the threshold value for p, player B will be choosing a strategy that minimizes the payoff to player A.)

In parts e) to g), we assume that player A announces her strategy first and that player B decides what to do based on this. A random strategy for player A consists of specifying the probability q that player A will accept a double if B wishes to double. For each hand received, player B can determine whether to double taking into account the hand and the value of q. As before, we assume that B always doubles when she has the six of hearts.

e) Suppose $q = 0$. Suppose B receives the 4 of hearts. Which results in a lower payoff to A: B doubling or B not doubling? What is the expected payoff of the game for player A under this scenario? (It will be a negative payoff).
f) Suppose $q = 1$. Suppose B receives the 4 of hearts. Which results in a lower payoff to A: B doubling or B not doubling? What is the expected payoff of the game for player A under this scenario?
g) Graph the payoff for player A as q goes from $q = 0$ to $q = 1$, assuming that B's doubling strategy gives the lowest payoff to A. There should be two lines graphed, one for cases when player B doubles with the four of hearts, and one for cases when B does not double with the four of hearts. What is the threshold value of q such that B is indifferent between doubling or not with the four of hearts?

(HINT: The optimal payoff in part g should be the same as the optimal payoff in part d.)

You may wish to play S-poker with a friend to get a sense for the "strategies." To simulate the probabilities, you can play an equivalent game in which A receives the 5 of hearts, and B receives the 2, 3, 4, 6, or 7 of hearts, each with probability 1/5.

Remark: "Bluffing" in this game refers to the situation when player B doubles the bet when she has a four of hearts, and thus a losing hand. This problem illustrates the importance of a randomized strategy for bluffing. B loses on average if she never bluffs, and she loses much more if she always bluffs. But if she bluffs the optimal amount of time, she is guaranteed to win on average. It also illustrates that one should not accept all doubles, but should accept a random proportion of the time. Player A loses a lot if she always accepts doubles, and loses a lot more if she never accepts doubles. She loses only a small amount if she accepts doubles a small random percentage of the time, assuming that player B is bluffing optimally.

6. Data Envelopment Analysis (10 points)
Pleasantville Bank has three branches. You have been asked to evaluate the efficiency of each. The following inputs are to be used for the study:

Input 1: Labor hours used (hundreds per month)
Input 2: Space used (hundreds of square feet)
Input 3: Supplies used per month (in dollars)

The outputs are defined as:

Output 1: Loan applications per month
Output 2: Deposits processed per month (in thousands)
Output 3: Checks processed per month (in thousands)

The relevant information for the banks is given in the following table:

		Inputs			Outputs	
Bank	1	2	3	1	2	3
1	14	23	51	220	18	45
2	15	20	50	200	15	35
3	16	19	51	210	17	30

a) Formulate three linear programs, one for testing the efficiency of each bank.
Let w_1, w_2, w_3 be the variables associated with the inputs, and t_1, t_2, t_3 the variables associated with the outputs. As in class, you should constrain each of the variables to be at least .00001.
b) Using ps5.xls, solve the three linear programs from part a). (You will have to modify the same spreadsheet 3 times.) Are any of the branches inefficient? If so, which ones? (You do not have to turn in the Excel output here—we only are interested in the value of the optimal solution.)

Challenge Problem E (8 points)
In the first example we looked at in class (slides 8-20), if the row player uses an optimal strategy then the expected payoff of each column is 1/9. Based on this, you might be tempted to think that in any payoff matrix, the expected payoff of each column is the same when the row player uses an optimal strategy. However, this is not true in general!

Find an example of a 2 x 2 payoff matrix where the expected value of each column is different when the row player uses an optimal strategy.

Hint for Problem 5 – Poker and Game Theory

It helps to read this problem over more than once, so you have a feel for what is going on. After you have a feel for what is going on, it is helpful to list the possible outcomes and probabilities in a table, such as this:

	Probability	Double	Accept	Payoff to A	Payoff to B
B = 4	3/5	N	-	1	-1
		Y	Y	2	-2
		Y	N	-1	1
B = 6	2/5	N	-	-1	1
		Y	Y	-2	2
		Y	N	-1	1

Now when you go to answer the questions, you can refer to the above table.

For instance, in part a), we assume that player B never doubles when receiving a 4. So, if B doubles, this means B has a 6. We want to know what is the payoff to player A if A accepts a double. You should just be able to read this off the table, under the entry 'B=6, A accepts, payoff to A'. You can find the payoff when A rejects a double similarly.

For the expected payoff of the game to player A, we assume that A chooses the better of the two strategies (accepting all doubles or rejecting all doubles). The expected payoff is then

(prob. B gets 4)(exp. payoff to A if B gets 4) + (prob. B gets 6)(exp. payoff to A if B gets 6)

under the strategy A has chosen.

Parts b) and c) are similar, only you may have to consider slightly more cases.