15.053 March 6, 2007

Duality 1: Pricing

Warning: heavier than usual theory ahead.

Quotes of the Day

What is a cynic? A man who knows the price of everything and the value of nothing.

Oscar Wilde

It's not enough to create magic. You have to create a price for magic, too. You have to create rules.

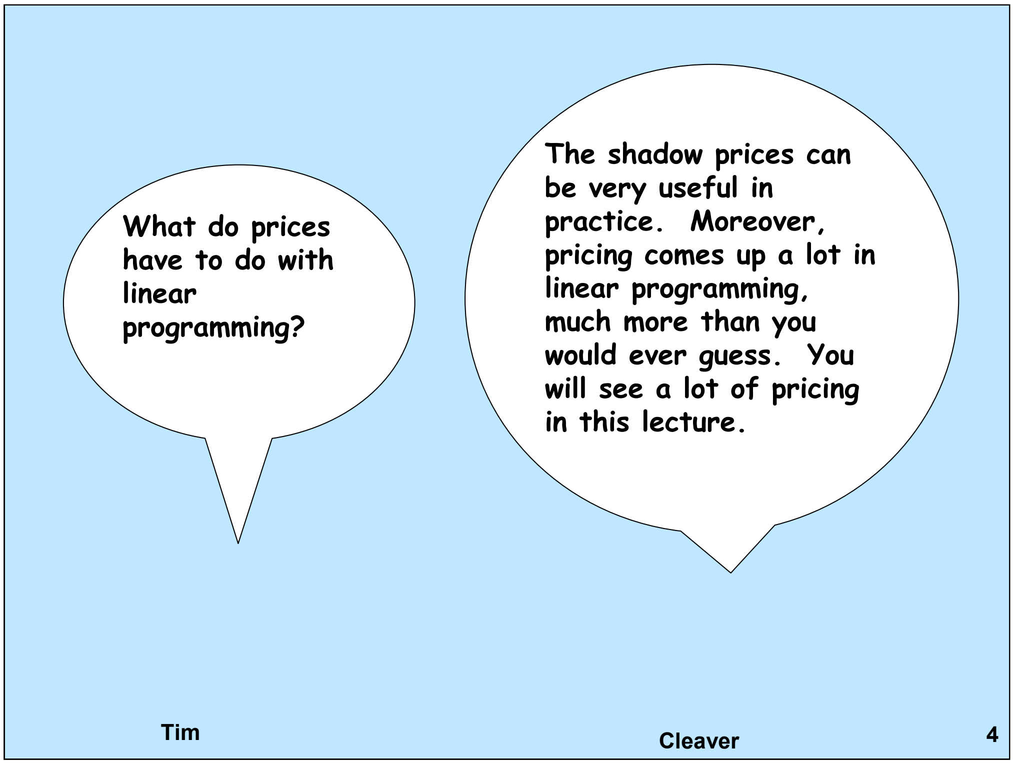
Eric Burns

The price of freedom is eternal vigilance.

Thomas Jefferson

Prices as Part of Linear Programming

- Shadow Prices
- Reduced costs and pricing out
- Prices as part of the simplex algorithm
- Prices because they are useful in practice



Normally, we introduce duality in linear programming directly. Duality involves prices and pricing out. But in this lecture, we will not talk about duality at all. We will talk about prices, especially shadow prices and simplex multipliers.

Example: Transfer Prices within a Firm

- **From Wikipedia: Transfer pricing refers to the pricing of goods and services within a multi-divisional organization.**

- **Example, goods from the production division may be sold to the marketing division, or goods from a parent company may be sold to a foreign subsidiary, with the choice of the transfer price affecting the division of the total profit among the parts of the company.**

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There are a lot of applications of pricing in practice. This is obvious as we are constantly paying prices for goods. What is less obvious is that prices are often used in more subtle ways. For example, firms use transfer prices in order to price goods between divisions. In principle, the transfer prices can be used to increase efficiency within a firm. In fact, linear programs can help determine transfer costs for many firms.

More likely, it will be used to take advantage of different tax codes in different countries. The way to save taxes is to have the profit of a firm show up in locations with low taxes, and to have no profit in those locations with a high tax rate.

Mira and Marnie's M&M Adventure

Mira and Marnie, two MIT undergraduates known as the M&M sisters, recently received a gift from their parents of 2000 pounds of gray M&Ms and 6000 pounds of red M&Ms, the MIT colors. So, they decided to go into business selling large bags of "MIT M&Ms" for frat parties. They can sell a bag with 3 pounds of red M&Ms and 2 pounds of gray M&Ms for $20. They can purchase bags of 3 pounds of red M&Ms and 4 pounds of gray M&Ms for $30. How many bags should Mira & Marnie buy and sell to maximize their profit.

This is the M&M adventure from previous lectures.

Formulation as a linear program

- Let x_1 be the number of 7 pound bags purchased (in thousands)
- Let x_2 be the number of 5 pound bags sold (in thousands)
- Measure the profit in $10,000s.

Prices for a linear program

Prices

max	$z =$	$-3x_1 + 2x_2$			
s.t.		$-3x_1 + 3x_2 \leq 6$	red		-.33
		$-4x_1 + 2x_2 \leq 2$	gray		17.2
		$x_1 \geq 0, \; x_2 \geq 0$			

A set of *prices* for a linear program is a collection of real numbers associated with each constraint, other than the nonnegativity constraints.

We will permit any type of prices with any signs. In the second lecture, we will sometimes require prices to be nonnegative or nonpositive.

A quick review of shadow prices

$$\begin{aligned} \text{max } z = & -3x_1 + 2x_2 \\ \text{s.t. } & -3x_1 + 3x_2 + x_3 = 6 \\ & -4x_1 + 2x_2 + x_4 = 2 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \end{aligned}$$

Shadow prices

1/3
1/2

Suppose that the number of red M&Ms on hand decreased from 6 thousand to 5.6 thousand. What is the impact on the optimal objective value?

z "increases" by $-.4 \times 1/3 = -4/30$.

Suppose that the number of gray M&Ms on hand decreased from 2 thousand to 1.7 thousand. What is the impact on the optimal objective value?

This depends on the range of the shadow price staying valid.

Quick Review of Shadow Prices

If the prices are the shadow prices, then the modified costs are the reduced costs for the optimal solution.

$$\max\ z = -3x_1 + 2x_2$$

$$\text{s.t.}\quad -3x_1 + 3x_2 + x_3 = 6$$

$$-4x_1 + 2x_2 + x_4 = 2$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Shadow prices
1/3
1/2

What happens to the optimum objective value if we increase the 6 to a 7?

In the lecture on sensitivity analysis, we priced out using the shadow prices. This is a classic use of shadow prices. In order to distinguish these reduced costs from other reduced costs, we refer to them as “optimal reduced costs”. It is always true that the optimal reduced costs of the basic variables are 0.

The optimal reduced cost for x_2 is $2 – (1/3)3 – (1/2)2 = 0$.

One definition of the optimal reduced cost is the following. If we increase the original cost by the negative of the reduced cost, then the variable is profitable enough to be in some optimal basis. For a basic variable, the reduced cost is 0, since the basic variables are already in an optimal basis.

Pricing out to get reduced costs

- For a maximization problem, treat the prices as though they really are prices on the RHS.

max	$z =$	$-3x_1 + 2x_2$		
s.t.		$-3x_1 + 3x_2 \leq 6$	red	1
		$-4x_1 + 2x_2 \leq 2$	gray	2
		$x_1 \geq 0, \; x_2 \geq 0$		

Prices: 1 (red), 2 (gray)

Reduced costs **= original costs minus column coefficients time prices.**

x_1	x_2
- 3	2
- (1 × -3)	- (1 × 3)
- (2 × -4)	- (2 × 2)
8	-5

Whenever there is a set of prices, we can compute "reduced costs" by pricing out. These reduced costs are not exactly the same as the ones given by the Excel sensitivity report. Those are the reduced costs when prices are the shadow prices, as we shall soon see. So, "reduced costs" are a much more general concept than the ones given in the Excel sensitivity report.

To compute the reduced cost of variable x_j, we start with the original cost coefficient of x_j. Then for each constraint, we subtract the price of that constraint times the column coefficient for variable x_j in that constraint. We illustrated with two examples on this slide.

Pricing out to get reduced costs

- For a maximization problem, treat the prices as though they really are prices on the RHS.

			Prices	
max	$z = -3x_1 + 2x_2$			
s.t.	$-3x_1 + 3x_2$	≤ 6	1	p_1
	$-4x_1 + 2x_2$	≤ 2	2	p_2
	$x_1 \geq 0,\ x_2 \geq 0$			

Reduced costs = original costs minus column coefficients time prices.

x_1

$$\begin{array}{r} -3 \\ -(1 \times -3) \\ -(2 \times -4) \\ \hline 8 \end{array}$$

$$c_j - \sum_i a_{ij} p_i$$

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Whenever there is a set of prices, we can compute "reduced costs" by pricing out. These reduced costs are not exactly the same as the ones given by the Excel sensitivity report. Those are the reduced costs when prices are the shadow prices, as we shall soon see. So, "reduced costs" are a much more general concept than the ones given in the Excel sensitivity report.

To compute the reduced cost of variable x_j, we start with the original cost coefficient of x_j. Then for each constraint, we subtract the price of that constraint times the column coefficient for variable x_j in that constraint. We illustrated with two examples on this slide.

Pricing out to get reduced costs

- Every vector of prices can be used to in pricing out. Each leads to a set of reduced costs.

max	z =	$-3x_1 + 2x_2$		
s.t.		$-3x_1 + 3x_2$	≤ 6	red
		$-4x_1 + 2x_2$	≤ 2	gray
		$x_1 \geq 0,\ x_2 \geq 0$		

Prices

red	-1
gray	3

x_1	x_2
- 3	
- (-1 × -3)	
- (3 × - 4)	
6	

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This slide is just to give you some practice. The reduced cost of x_2 is 2 – (-1)(3) – 3(2) = -1.

When Prices are Shadow Prices

If the prices are the shadow prices, then the modified costs are the reduced costs for the optimal solution.

max	$z =$	$-3x_1 + 2x_2$		
s.t.		$-3x_1 + 3x_2 + x_3$		$= 6$
		$-4x_1 + 2x_2$	$+ x_4$	$= 2$
		$x_1 \geq 0, \; x_2 \geq 0$		

Shadow prices

1/3
1/2

The optimal solution:

$x_1 = 1, \; x_2 = 3,$
$x_3 = 0, \; x_4 = 0$

x_1	x_2
$- 3$	
$- (1/3 \times -3)$	
$- (1/2 \times -4)$	
0	

The *optimal reduced costs* of basic variables are 0.

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In the lecture on sensitivity analysis, we priced out using the shadow prices. This is a classic use of shadow prices. In order to distinguish these reduced costs from other reduced costs, we refer to them as "optimal reduced costs". It is always true that the optimal reduced costs of the basic variables are 0.

The optimal reduced cost for x_2 is $2 – (1/3)3 – (1/2)2 = 0$.

One definition of the optimal reduced cost is the following. If we increase the original cost by the negative of the reduced cost, then the variable is profitable enough to be in some optimal basis. For a basic variable, the reduced cost is 0, since the basic variables are already in an optimal basis.

When Prices are Shadow Prices

The optimal reduced costs for non-basic variables (of a max problem in standard form) are ≤ 0.

$$\begin{aligned} \max\ z = \ & -3x_1 + 2x_2 \\ \text{s.t.}\ \ & -3x_1 + 3x_2 + x_3 = 6 \\ & -4x_1 + 2x_2 + x_4 = 2 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Shadow prices

Shadow prices
1/3
1/2

The optimal solution:

$x_1 = 1,\ x_2 = 3,$
$x_3 = 0,\ x_4 = 0$
$z = 3.$

x_3	x_4
0 - (1/3 × 1) - (1/2 × 0) = -1/3	

$$0 - (1/3 \times 1) - (1/2 \times 0) = -1/3$$

We will see why shortly.

For a maximization problem, the reduced costs of nonbasic variables are always ≤ 0. If it were positive, then the nonbasic variable would be profitable enough to be in an optimal basis.

Optimal Reduced Costs

- If the reduced cost of a variable is -1/3 ($\bar{c}_j$) then increasing the original cost by more than 1/3 (- $\bar{c}_j$) will make it profitable to bring it into the basis (assuming non-degeneracy)
- If we increase it by exactly 1/3, then there will be alternative optimum solutions.

On Prices, Shadow Prices, and Reduced Costs

Prices: a value for each constraint.

Shadow price. The rate of increase in the optimal objective value per unit of increase in the RHS. (A type of price).

Reduced cost. The value obtained by using the pricing out formula using prices.

$$c_j - \sum_i a_{ij} p_i$$

Optimal Reduced Costs. If $\bar{c}_j$ is the optimal reduced cost, then increasing the cost of x_j by more than $\bar{c}_j$ will make variable x_j profitable; that is, x_j will be in some optimum bfs.

Pricing out a new variable

- Mira and Marnie decide that they can sell different sized bags for more money. If they sell bags with 4 pounds of red M&Ms and 3 pounds of gray M&Ms, they can charge $30. Will it be profitable for them to do so?

- Suppose that there is a request for as many gray M&Ms as possible. How many gray M&Ms can be included with 4 pounds of red M&Ms and sold for $30 so that the bags are profitable?

Pricing out a new variable

If we introduce a new variable, we price it out just as we priced out all other variables.

$$\max\ z = -3x_1 + 2x_2 + 3y$$

$$\text{s.t.}\quad -3x_1 + 3x_2 + 4y + x_3 = 6$$

$$-4x_1 + 2x_2 + 3y + x_4 = 2$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ y \geq 0$$

Shadow prices

Shadow prices
1/3
1/2

y

$$\begin{array}{r} 3 \\ -\ (1/3 \times 4) \\ -\ (1/2 \times 3) \\ \hline 1/6 \end{array}$$

If we price out a new variable, and it is its reduced cost is positive, then the opt objective value will increase if the variable can be positive.

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We can price out a new variable using the shadow prices. By new variable, we mean a variable that was not in the original problem that we optimized.

Original problem. The problem without the variable y.

New problem. The original problem with the variable y added to it.

The shadow prices were for the original problem.

The optimal solution for the original problem was $x_1 = 1$, $x_2 = 3$, $x_3 = 0$, $z = 3$. If the new variable y prices out to be non-positive, then the same solution (with y = 0) will also be optimal for the new problem. However, if the new variable prices out to be positive, then the old solution will not be optimal for the new problem (assuming non-degeneracy) because it would be profitable to have $y > 0$.

The reduced cost for the new variable is 3 – (1/3)(4) – (1/2)(3) = 1/6. So it is profitable to sell M&Ms in the new bag size.

If we let g be the number of gray M&Ms in the bag, then the reduced cost is 3 – (1/3)(4) – (1/2)(g) = 0. We use 0 so that g gives the breakeven amount. Then g = 2[3 – (1/3)(4)] = 3 1/3 pounds.

The Gray Panthers Stage a Protest

- **Mira and Marnie are told that they need to be more fair to gray M&Ms. The gray panthers have insisted that the number of gray M&Ms must be at least as great as the number of red M&Ms.**

Mira and Marnie consider selling bags with 4 pounds of red M&Ms and 4 pounds of gray M&Ms for $30. Will it be profitable? What would it cost them in lost profit to sell 5 bags?

The gray panthers usually focus on issues more closely related to social security, Medicare, and age discrimination. But who knows? Perhaps they are also concerned about prejudice against the color gray.

The effect of gray fairness

3 - 4 × 1/3 - 4 × ½ = -1/3.

Units: in 1000s of bags

Profit in $10s of thousands of dollars

Selling one thousand bags will cost Mira and Marnie 1/3 × $10,000 = $3,333.33

Selling one bag will cost Mira and Marnie $3.33

Selling 5 bags will cost them 5 × $3.33 = $16.65 (assuming that the range is valid for this change).

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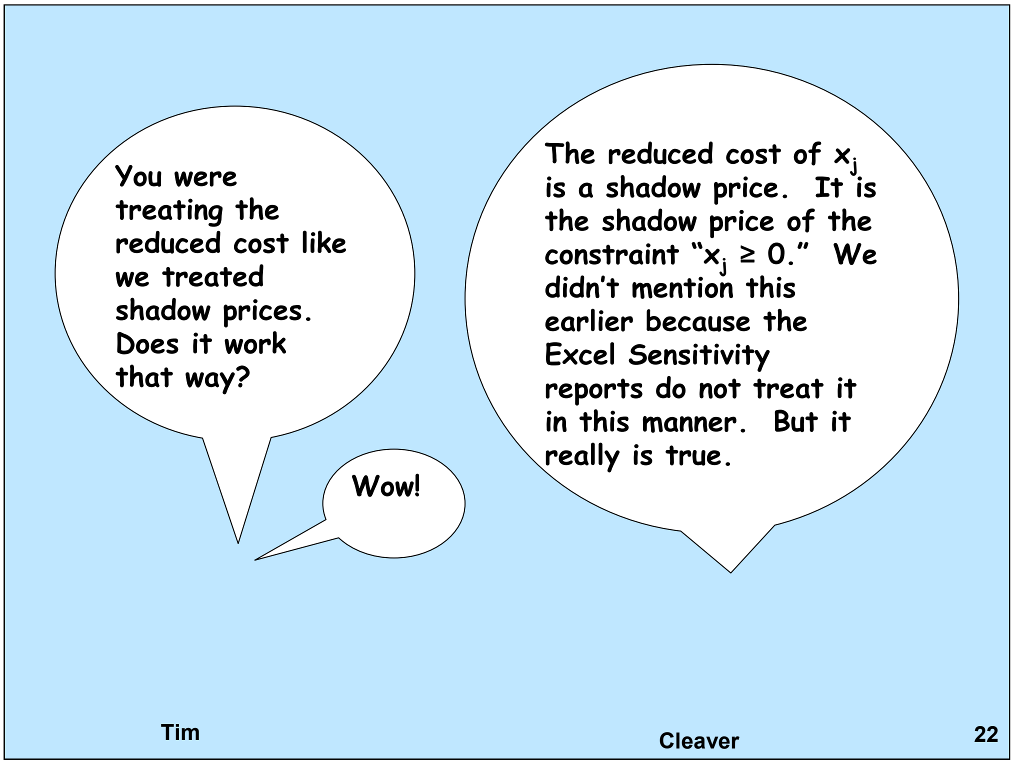
One complication here is to keep the units straight. We may want to measure the impact of changes in dollars. But the objective is currently written in 10s of thousands of dollars, and the variables are measured in units of thousands of bags. The first thing to do is to remember that the units may be different, and then use techniques you learned in science classes to make sure that you interpret shadow prices correctly by correctly performing any conversions of variables.



We said that we don't compute prices for the nonnegativity constraints. This is true when we are pricing out. However, the shadow price is well defined as the unit increase in the optimal objective function per unit increase in the RHS coefficient for the constraint. And here we point out that the shadow price is the same as the reduced cost.

Theorem

- **The reduced cost of variable xj is the shadow price of the constraint "$x_j \geq 0$." That is, the reduced cost is the "increase" in the objective function if we add the constraint "$x_j \geq 1$."**
- **Excel does not provide the range for the validity of this shadow price. In fact, it doesn't even label it as a shadow price at all.**
- **Note. The reduced costs of the basic variables are 0. And typically, these variables are positive, and so there is no change in the opt objective for adding the constraint "$x_j \geq 1$" except that possibly we are outside the range of the shadow price.**

As usual, we cannot guarantee that the shadow price is valid in a large interval. So, we really mean it is the rate of change in the optimal objective function per unit change in the RHS of the constraint "$x_j \geq 0$."

Excel is not to fault here. I'm not aware of any program that computes ranges on the shadow prices of the nonnegativity constraints.

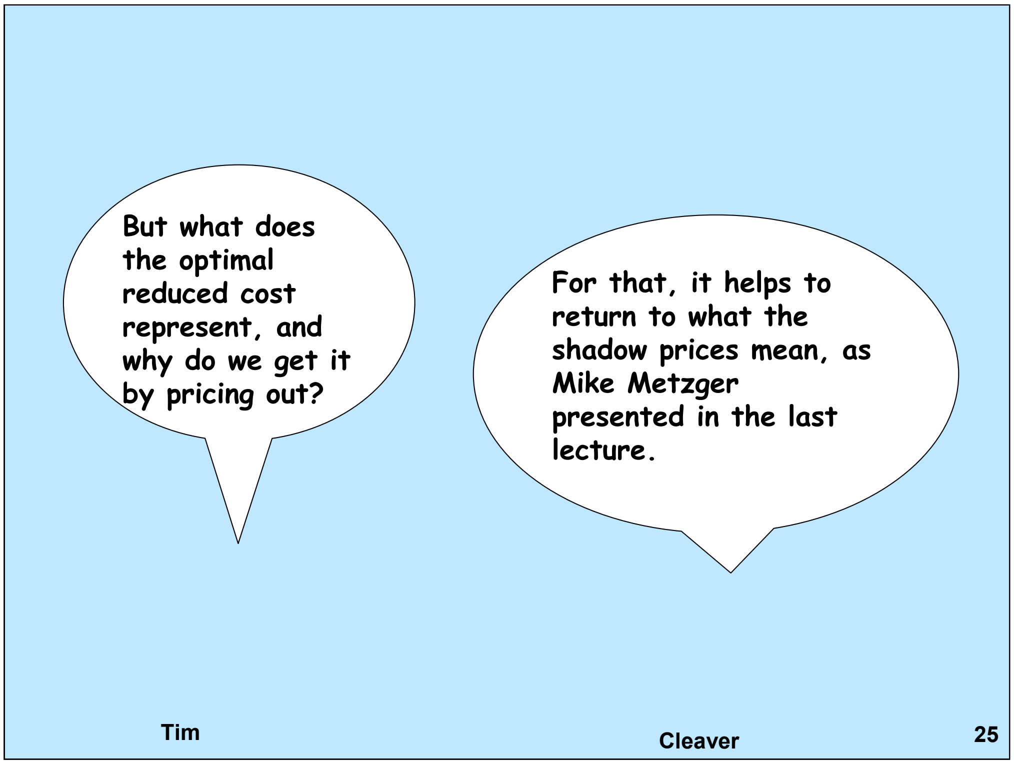
The range given by the Excel program for variable xj really is the reduced cost. If the reduced cost is -5, then Excel will report that the variable x_j is not in any optimal solution if its cost coefficient is reduced. And it will be in some optimal solution if its cost is increased by 5.

The range on the shadow price is different. We know that the optimal objective value goes down by 5 if we require that $x_j \geq 1$. And it will go down by 5q if we require that $x_j \geq q$, so long as q is not too large. What we don't know is how large q can be.

There is a price given for every constraint except nonnegativity. Then we can obtain reduced costs by pricing out.

If the prices are the shadow prices, then we obtain the optimum reduced costs, the ones given by the Excel sensitivity report. These are also the “shadow prices” of the nonnegativity constraints.

The M&M brothers help summarize the lecture so far.



These next slides were written earlier, and I have hidden them from the lecture. You can check them out if you want.

A quick review of shadow prices

$$\begin{array}{llr} \max & z = -3x_1 + 2x_2 & \\ \text{s.t.} & -3x_1 + 3x_2 + x_3 & = 6 \\ & -4x_1 + 2x_2 + x_4 & = 2 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 & \end{array}$$

Shadow prices

1/3
1/2

Suppose that the number of red M&Ms on hand decreased from 6 thousand to 5.6 thousand. What is the impact on the optimal objective value?

z "increases" by $-.4 \times 1/3 = -4/30$.

Suppose that the number of gray M&Ms on hand decreased from 2 thousand to 1.7 thousand. What is the impact on the optimal objective value?

This depends on the range of the shadow price staying valid.

Shadow prices and reduced costs

$$\max\ z = -3x_1 + 2x_2 + 2.5y$$

s.t.

$$-3x_1 + 3x_2 + 4y + x_3 = 6$$

$$-4x_1 + 2x_2 + 3y + x_4 = 2$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ y \geq 0$$

Shadow prices

1/3
1/2

What is the impact of setting y to .01 and optimizing with respect to the remaining variables?

"increase" in the optimum objective value

$2.5 \times .01$ (direct profit)

$- 1/3 \times (4 \times .01)$ (due to red M&Ms)

$- 1/2 \times (3 \times .01)$ (due to gray M&Ms)

$= -1/3 \times .01 = -1/300.$

2.5
- 1 1/3
- 1.5

= -1/3

This depends on the range of the shadow prices staying valid.

A comment on the range

The change in costs might not be valid if we set y = .01.

But except under degeneracy, it will be valid if we set y small enough, e.g., .0001.

Let f(q) be the optimal objective obtained by setting y to q and optimizing the other variables.

$f(0) = 3$	the original optimum objective value.
$f'(0) = -1/3$	the reduced cost for y in an optimal solution.

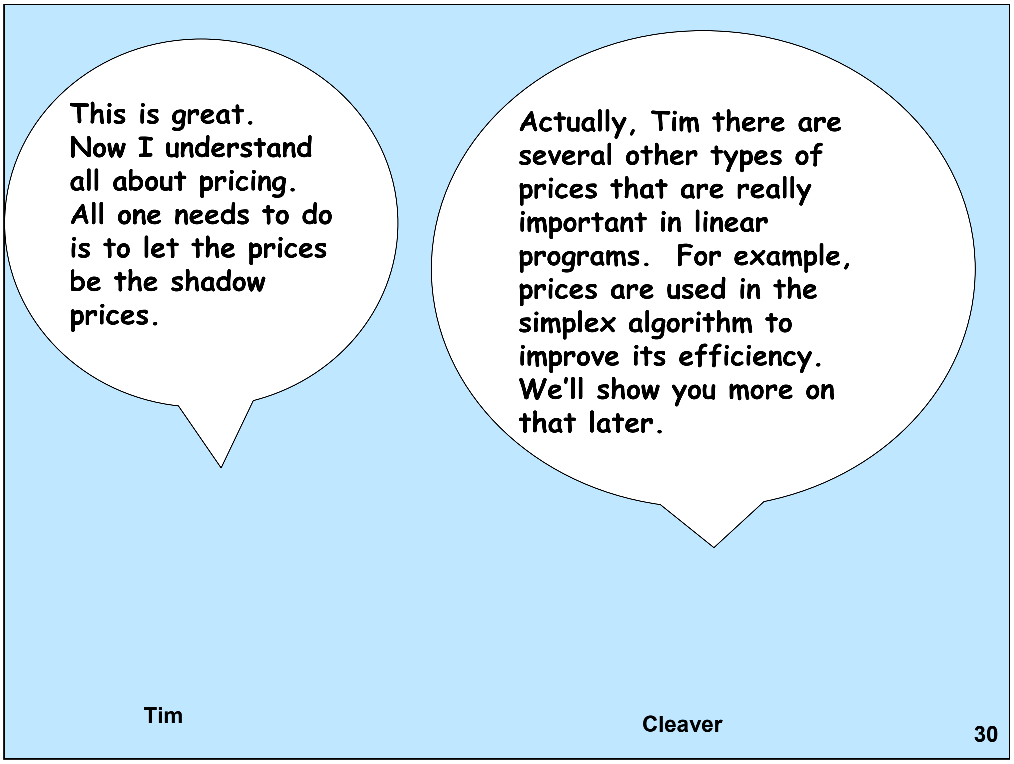
Making the new bags of M&Ms profitable

$$\begin{aligned}
\text{max } z &= -3x_1 + 2x_2 + 2.5y \\
\text{s.t. } & -3x_1 + 3x_2 + 4y + x_3 = 6 \\
& -4x_1 + 2x_2 + 3y + x_4 = 2 \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ y \geq 0
\end{aligned}$$

Shadow prices

1/3
1/2

The variable y is the number of thousands of bags sold with 4 pounds of red M&Ms and 3 pounds of gray M&Ms. How much do we need to increase the profitability of a bag for it to become profitable to sell these bags?



Obtaining the optimal reduced costs from shadow prices is a great application of pricing out. But it is far from the only application. there are three ways in which pricing out is used, and these ways interrelate.

1. Sensitivity analysis
2. Algorithmic pricing, that is pricing in the context of carrying out an algorithm in order to increase algorithmic efficiencies. (In two lectures, we will use prices for the "dual simplex algorithm.")
3. Duality (to be explained next lecture).

Mental Break

- **The MacVicar Fellowship**

Prices for the simplex algorithm

z	x_1	x_2	x_3	x_4		
-1	-3	+2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

$z = -3x_1 + 2x_2$

$-z - 3x_1 + 2x_2 = 0$

For this half of the lecture, we will multiply the z-row by -1.

In this way, the original coefficients in the z-row are the real cost coefficients.

Next year, we will have all of our z-rows in this form because it requires less multiplications by -1. But using this form isn't perfect either. It's just advantageous.

On Modifying the z-row

z	x_1	x_2	x_3	x_4		RHS	Prices
-1	1	0	0	-1	=	-2	
0	-3	3	1	0	=	6	0
0	-4	2	0	1	=	2	1

Pricing out means that we subtract constraint 2 from the z-row

This does not change the LP.

The feasible region is the same. So is the set of optimal solutions.

Adding multiples of equality constraints to the z-row does not change the feasible region nor does it change what solutions are optimal. But it does change the z-row. The simplex algorithm chooses prices so that when we price out, we obtain a z-row that has properties that we want.

Any vector of prices can lead to pricing out.

z	x_1	x_2	x_3	x_4			Prices
-1	-4	1	-1	1	=	-4	
0	-3	3	1	0	=	6	1
0	-4	2	0	1	=	2	-1

Here we subtract constraint 1 from the z-row and we add constraint 2 to the z-row.

This is some practice on pricing out in the simplex algorithm.

The simplex algorithm chooses prices so that the reduced costs of basic variables is 0.

z	x_1	x_2	x_3	x_4			Prices
-1	-3	+2	0	0	=	0	
0	-3	3	1	0	=	6	0
0	-4	2	0	1	=	2	0

Here we start with the costs of the basic variables as 0.

The simplex algorithm chooses prices of 0.

We generally don't think of the simplex algorithm as "choosing prices." We think of it as carrying out a sequence of pivots that correspond to moving along edges of the feasible region.

But the simplex algorithm always carries out pivots so that the coefficients of the basic variables of the z-row are all 0. The reason that this is advantageous is that it permits us to identify when a tableau is optimal, that is, all other z-row coefficients are ≥ 0 if we use our usual representation of the z-row. And it permits us to identify an entering variable when its coefficient is negative. These rules only work because the coefficients of the basic variables are 0.

Interestingly, we can focus entirely on the coefficients in the z-row, and realize that they may be obtained by pricing out. The prices are the unique values that can be obtained so that the coefficients of the basic variables are 0.

z	x_1	x_2	x_3	x_4			Prices after one pivot
-1	-3	+2	0	0	=	0	
0	-3	3	1	0	=	6	0
0	-4	2	0	1	=	2	1

z	x_1	x_2	x_3	x_4		
-1	1	0	0	-1	=	-2
0	-3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

The tableau after one pivot.

This is another example showing that the z-row can be obtained by pricing out.

Prices after two pivots

z	x_1	x_2	x_3	x_4		RHS	Prices
-1	-3	+2	0	0	=	0	
0	-3	3	1	0	=	6	1/3
0	-4	2	0	1	=	2	1/2

The tableau after two pivots.

z	x_1	x_2	x_3	x_4		RHS
-1	0	0	-1/3	-1/2	=	-3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	1/2	=	3

This is another example showing that the z-row in the "current tableau" can be obtained by pricing out.

Tim is quite right that the importance of simple multipliers needs to be explained. In fact, simplex multipliers are closely linked to shadow prices (in a surprising way), and they are very useful in and of themselves.

Simplex Multipliers

Start with an original problem

Obtain a modified problem and modified tableau after several pivots.

The ***simplex multipliers*** are those prices such that the reduced costs wrt to those prices are exactly the same as the reduced costs in the modified tableau. The basic variables have a reduced cost of 0.

Reduced costs

0	0	-1/3	-1/2		-3

Simplex Multipliers

1/3
1/2

Go back two slides.

More on prices and reduced costs

Prices: **a value for each constraint.**

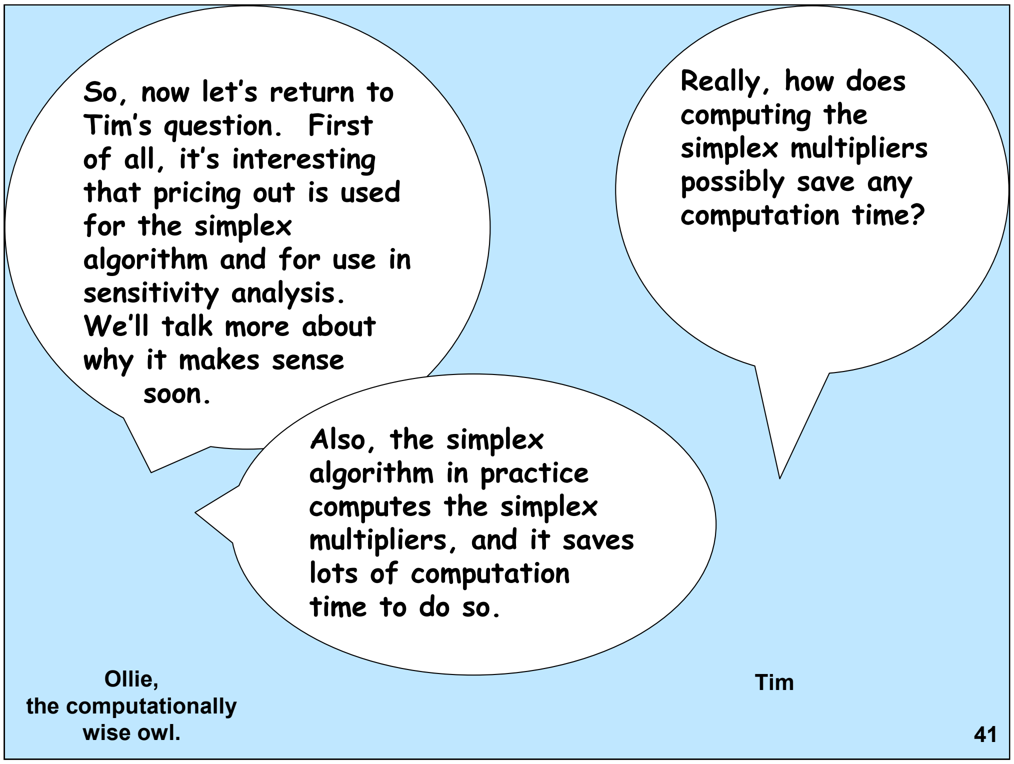
Shadow price**. The rate of increase in the optimal objective value per unit of increase in the RHS. (A type of price).**

Simplex multiplier. **A kind of price used in the simplex algorithm**

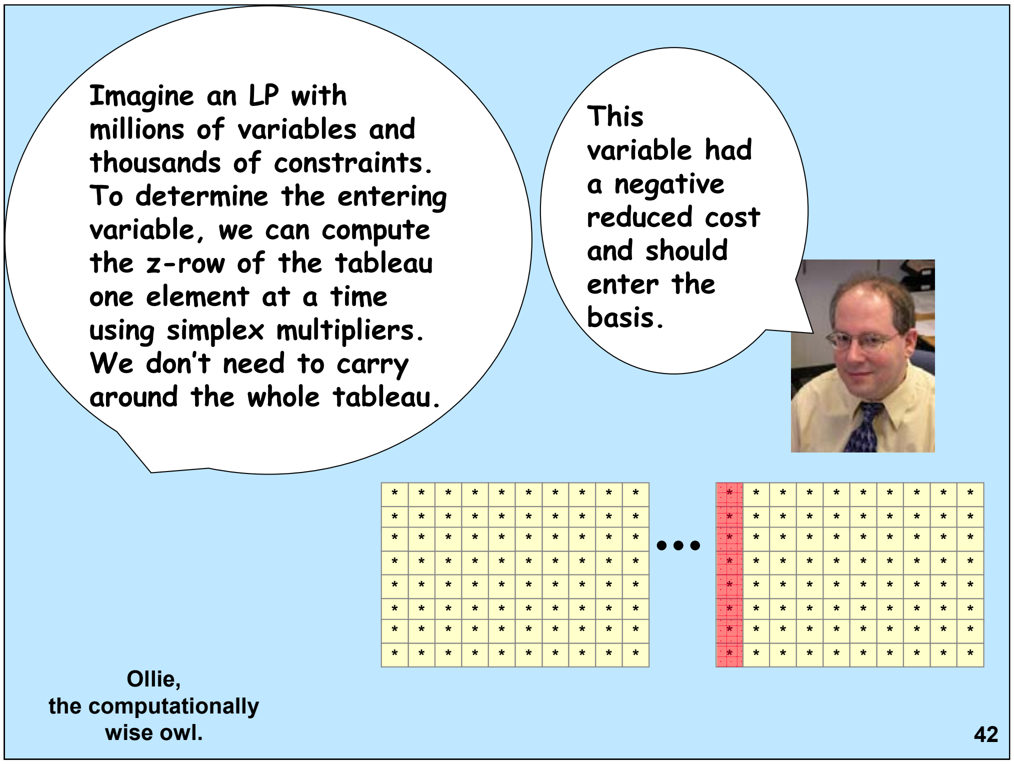
Reduced cost**. The value obtained by using the pricing out formula using prices.**

Optimal Reduced Costs. **The shadow prices of the non-negativity constraints**

Reduced costs using simplex multipliers. **The values in the z-row of a tableau. We use these costs in the simplex algorithm to determine optimality and determine what variable should enter the basis.**



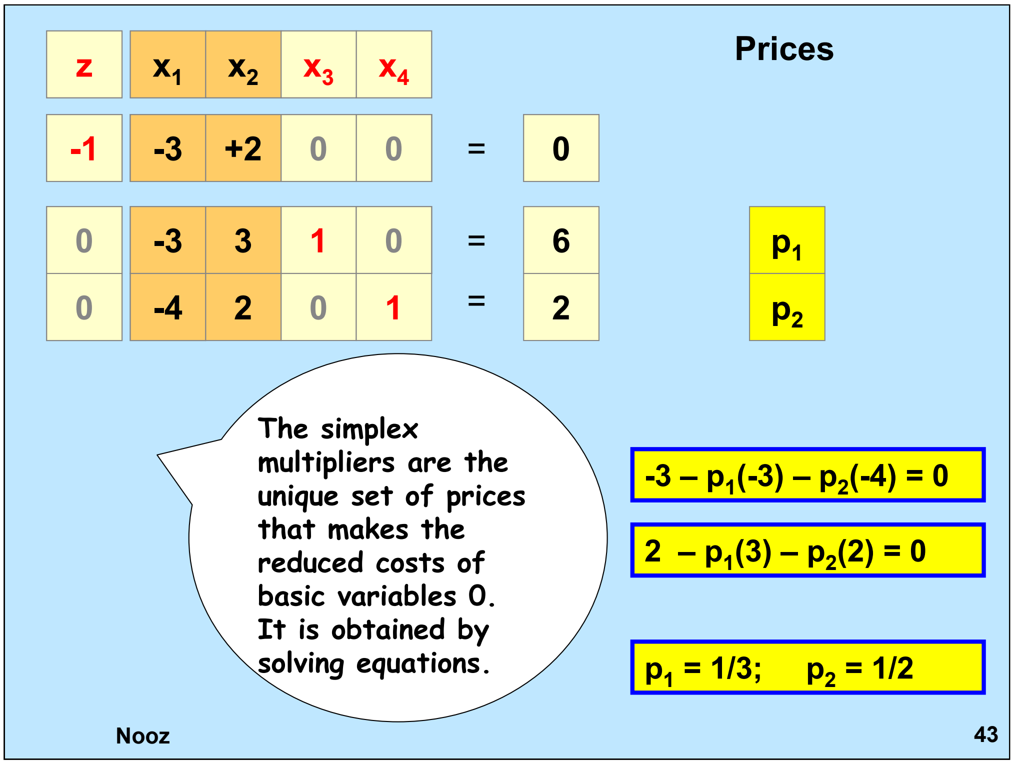
The idea of saving computation time by using simplex multipliers at first makes no sense. After all, if we carry out the simplex algorithm using tableaus, then where is there any savings? We shall see why on the next slide.



In reality, lots of linear programs have many more variables than constraints. So, a problem may have a million variables and only 10,000 constraints. Moreover, many of these constraints may have a simplex multiplier that is 0, and this makes reduced costs even quicker to compute.

When we carry out the simplex algorithm is carried out, we need to identify a variable whose reduced cost is strictly positive. (In our usual representation of the z-row, the coefficients are the negatives of the reduced costs). We don't want to carry out pivots on all million variables. And the simplex algorithm in practice doesn't do that. Rather, it computes the simplex multipliers, and then it prices out nonbasic variables until it identifies a variable with positive reduced cost. This is much faster than storing all of the tableau.

We are deliberately leaving off what to do after obtaining an entering variable. Essentially one needs to recreate the column of the tableau for that variable. One can do this using linear algebra, but the details are beyond the scope of this lecture.

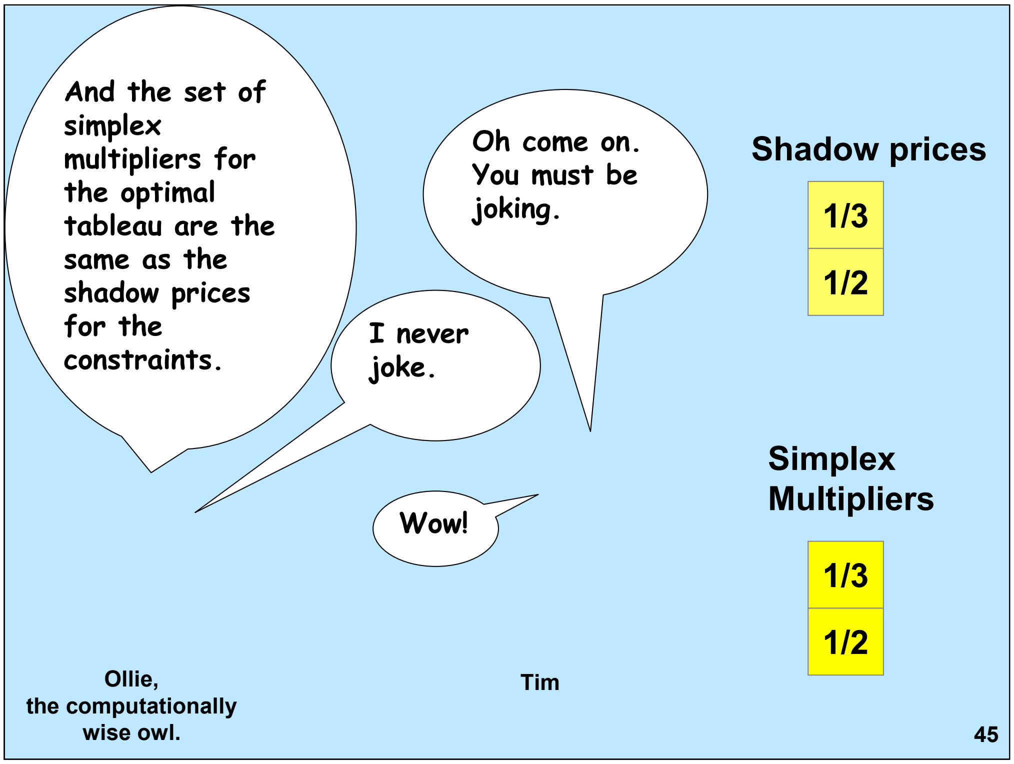


The simplex multipliers are chosen by requiring the reduced costs of the basic variables to be 0.

If we have m equality constraints, then there will be m basic variables. To determine the simplex multipliers, one solves m equations in m unknowns, obtaining a unique solution. (For those who have had 18.06, you know that we need to verify that the constraints are linearly independent. But the columns of the basic variables (ignoring the z-row) are linearly independent, and so its rows are too. If we used more linear algebra in the class, we would have established these two facts.)

More on pricing out in the simplex algorithm.

- The simplex multipliers are computed by solving a system of equations.
- They then can be used to price out each variable, without needing to store the entire tableau.
- If a variable is found to have a positive reduced cost, then will pivot in that variable.
- We are leaving out lots of details, such as how to create the column for that variable in the tableau so that we can do the min ratio test.



This should come as a complete mystery since the simplex multipliers are chosen specifically to make the reduced costs of the basic variables 0. They are not chosen to have anything to do with sensitivity analysis.

But on reflection, you may recall that when we price out with shadow prices, the reduced costs of the basic variables are 0. This could only have happened if the shadow prices are also the simplex multipliers.

z	x_1	x_2	x_3	x_4		
-1	-3	+2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

Prices after two pivots

1/3
1/2

z	x_1	x_2	x_3	x_4		
-1	0	0	-1/3	-1/2	=	-3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	1/2	=	3

The tableau after two pivots.

On the next slide, we will provide another perspective that explains why the simplex multipliers also give the shadow prices.

Shadow Prices and Opt Multipliers

- **What happens to the final z-row if we increase the RHS of constraint 1 from 6 to 7?**

z	x_1	x_2	x_3	x_4		RHS	opt multipliers
-1	-3	+2	0	0	=	0	
0	-3	3	1	0	=	7	1/3
0	-4	2	0	1	=	2	1/2
	0	0	-1/3	-1/2		-3 1/3	z-row in final tableau.

z increases by 1/3

If we change the RHS of the first constraint from 6 to 7, it has no impact on the simplex multipliers. They were chosen to make the reduced costs of x1 and x2 to be 0 in the final tableau.

But changing the 6 to a 7 does have an impact on the RHS coefficient of the z-row. In particular, it decreases it by 1/3, and thus increases the objective value by 1/3, which is the value of the simplex multiplier. By definition, the shadow price is the increase in the optimal objective value per unit increase in the RHS. So, the shadow price of constraint 1 must also be 1/3, and must equal the simplex multiplier.

Review of Results

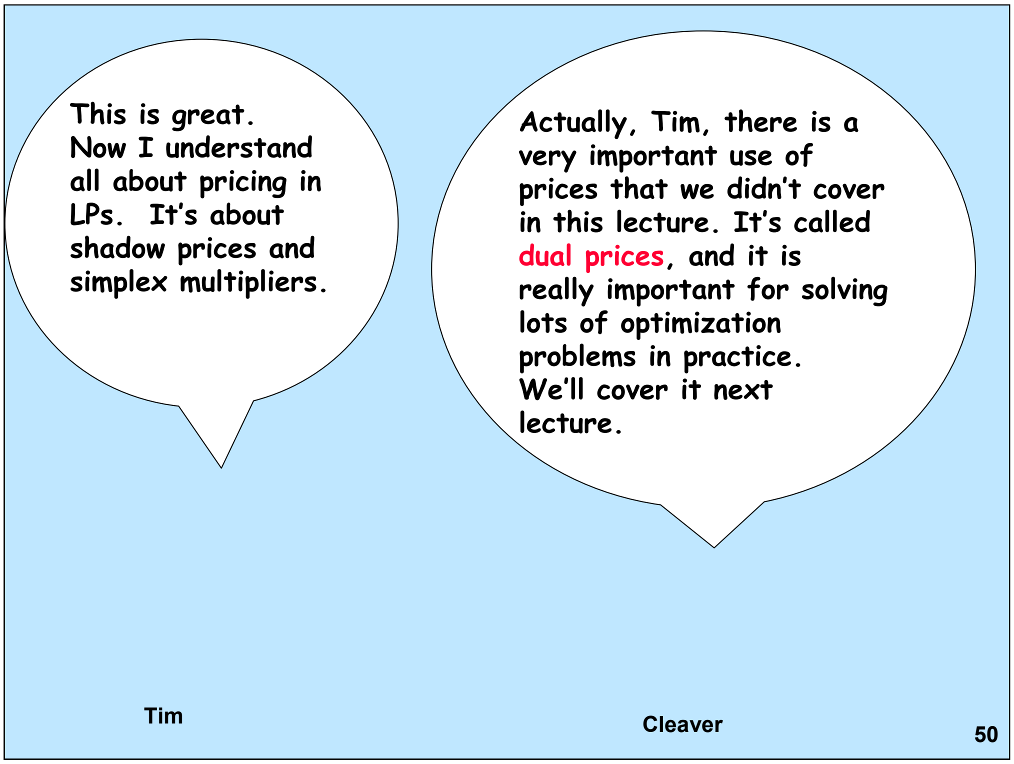
- The shadow prices are the unit change in the optimal objective value per unit change in the RHS coefficients.
- The simplex multipliers are chosen so that the reduced costs of the basic variables are 0.
- The simplex multipliers for the optimal tableau are equal to the shadow prices.
- The reduced costs of the nonbasic variables in the final tableau are the same as the reduced costs of the optimal solution.

IMPORTANT WARNING

z	x_1	x_2	x_3	x_4		
-1	0	0	-1/3	-1/2	=	-3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	1/2	=	3

In this lecture we multiplied the z-row by -1 so that we could work with original cost data and reduced costs. Normally, the z-row has the negative of the reduced costs. So be careful about the signs.

1	0	0	1/3	1/2	3



Tim and Cleaver will see you both next lecture when we cover dual prices and dual linear programs.

And now, it's time for …..