15.053 March 13, 2007

Duality 3

There are concepts much more difficult to grasp than duality in linear programming.

-- Jim Orlin

The concept [of nonduality], often described in English as "nondualism," is extremely hard for the mind to grasp or visualize, since the mind engages constantly in the making of distinctions and nondualism represents the rejection or transcendence of all distinctions.

from www.nonduality.com

Overview

- Rules for creating a dual linear program
- Complementary slackness conditions
- The dual simplex algorithm

Rules for creating a dual linear program

The Primal LP

		Dual Variable
Maximize	$z = 1x_1 + 2x_2 + 3x_3 + 4x_4$	
subject to	$8x_1 + 9x_2 + 10x_3 + 11x_4 \ ? \ 5$	y_1
	$12x_1 + 13x_2 + 14x_3 + 15x_4 \ ? \ 6$	y_2
	$16x_1 + 17x_2 + 18x_3 + 19x_4 \ ? \ 7$	y_3
	$x_1 \ ? \quad x_2 \ ? \quad x_3 \ ? \quad x_4 \ ?$	

The Dual LP

Minimize $v = 5y_1 + 6y_2 + 7y_3$

subject to

$$8y_1 + 12y_2 + 16y_3 \ ? \ 1$$
$$9y_1 + 13y_2 + 17y_3 \ ? \ 2$$
$$10y_1 + 14y_2 + 18y_3 \ ? \ 3$$
$$11y_1 + 15y_2 + 19y_3 \ ? \ 4$$
$$y_1 \ ? \quad y_2 \ ? \quad y_3 \ ?$$

The SOB approach
sensible, odd, and bizarre

The Primal LP

$$\text{Max} \quad z = 1x_1 + 2x_2 + 3x_3 + 4x_4$$

$$\begin{aligned} \text{s.t.} \quad & 8x_1 + 9x_2 + 10x_3 + 11x_4 \leq 5 && \text{S} \\ & 12x_1 + 13x_2 + 14x_3 + 15x_4 = 6 && \text{O} \\ & 16x_1 + 17x_2 + 18x_3 + 19x_4 \geq 7 && \text{B} \end{aligned}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \text{ uis}, \quad x_4 \leq 0 \ ?$$

S S O B

uis. unconstrained in sign

The SOB approach
sensible, odd, and bizarre

The Dual LP

Min	$v = 5y_1 + 6y_2 + 7y_3$			
s.t	$8y_1 + 12y_2 + 16y_3$	$\geq$	1	S
	$9y_1 + 13y_2 + 17y_3$	$\geq$	2	S
	$10y_1 + 14y_2 + 18y_3$	$=$	3	O
	$11y_1 + 15y_2 + 19y_3$	$\leq$	4	B

$y_1 \geq 0$, y_2 uis $y_3 \leq 0$

S O B

More Rules

Primal max	Dual min	
i-th constraint ≤	i-th variable ≥ 0	S
i-th constraint =	i-th variable free	O
i-th constraint ≥	i-th variable ≤ 0	B
i-th variable ≥ 0	i-th constraint ≥	S
i-th variable free	i-th constraint =	O
i-th variable ≤ 0	i-th constraint ≤	B

S: sensible

O: odd

B: bizarre

The SOB technique is due to Art Benjamin

The Primal LP

Maximize $z = 1x_1 + 2x_2 + 3x_3 + 4x_4$

subject to

$$
\begin{array}{lcl}
8x_1 + 9x_2 + 10x_3 + 11x_4 & \boxed{a} & 5 \\
12x_1 + 13x_2 + 14x_3 + 15x_4 & \boxed{b} & 6 \\
16x_1 + 17x_2 + 18x_3 + 19x_4 & \boxed{c} & 7
\end{array}
$$

$$x_1 \boxed{d} \quad x_2 \boxed{e} \quad x_3 \boxed{f}, \quad x_4 \boxed{g}$$

The Dual LP

Minimize $v = 5y_1 + 6y_2 + 7y_3$

subject to

$$
\begin{array}{lcl}
8y_1 + 12y_2 + 16y_3 & \boxed{d} & 1 \\
9y_1 + 13y_2 + 17y_3 & \boxed{e} & 2 \\
10y_1 + 14y_2 + 18y_3 & \boxed{f} & 3 \\
11y_1 + 15y_2 + 19y_3 & \boxed{g} & 4
\end{array}
$$

$$y_1 \boxed{a} \quad y_2 \boxed{b} \quad y_3 \boxed{c}$$

Shadow Price Method

The Primal LP

Maximize $z = 1x_1 + 2x_2 + 3x_3 + 4x_4$

subject to

$$8x_1 + 9x_2 + 10x_3 + 11x_4 \leq 5$$
$$12x_1 + 13x_2 + 14x_3 + 15x_4 = 6$$
$$16x_1 + 17x_2 + 18x_3 + 19x_4 \geq 7$$

x_1 [d] x_2 [e] x_3 [f], x_4 [g]

RHS of constraint (1) increases from 5 to 5.1 implies
Primal LP is less constrained implies
z increases or stays the same implies $y_1 \geq 0$.

RHS of constraint (3) increases from 7 to 7.3 implies
Primal LP is ________________ implies
z ____________________________ implies y_3 ____

Another Primal-Dual Pair

- **Jane Jetson is trying to design a diet for George on her home food dispenser so as to satisfy requirements and minimize cost.**

	Space Burgers	Saturn Shakes	Venus Flies	Requirements
Carbs	4	2	0	12
Fat	3	4	0	8
Protein	4	1	2	20
Cost (in $100s)	5	4	2	

Jane's Linear Program

x_1 = number of Spaceburgers

x_2 = number of Saturn Shakes

x_3 = number of Venus Flies

$$
\begin{aligned}
\text{minimize} \quad z = 5x_1 + 4x_2 + 2x_3 & \\
\text{subject to} \quad 4x_1 + 2x_2 + & \geq 12 \\
3x_1 + 4x_2 + & \geq 8 \\
4x_1 + 1x_2 + 2x_3 & \geq 20 \\
x_1, x_2, x_3 \geq 0 &
\end{aligned}
$$

Primal Linear Program

$$\begin{array}{llllll} \min & z = 5x_1 + 4x_2 + 2x_3 & & & & \\ \text{s.t} & 4x_1 + 2x_2 + & \geq 12 & \text{carbs} & & p_1 \\ & 3x_1 + 4x_2 + & \geq 8 & \text{fat} & & p_2 \\ & 4x_1 + 1x_2 + 2x_3 & \geq 20 & \text{protein} & & p_3 \\ & x_1, x_2, x_3 \geq 0 & & & & \end{array}$$

Dual Linear Program

$$\begin{array}{llll} \text{Max} & v = 12p_1 + 8p_2 + 20p_3 & & \text{cost of nutrients} \\ \text{s.t} & 4p_1 + 3p_2 + 4p_3 \leq 5 & & \text{burgers} \\ & 2p_1 + 4p_2 + 1p_3 \leq 4 & & \text{shakes} \\ & + 2p_3 \leq 2 & & \text{flies} \\ & p_1, p_2, p_3 \geq 0 & & \end{array}$$

Spacely Sprockets has decided to make food pellets. Mr. Spacely wants to Jane to buy all foods from him. How should he price them to maximize profit and so that Jane is willing to purchase all foods from him?

Dual Linear Program

Max	$z = 12\,p_1 + 8\,p_2 + 20\,p_3$	cost of pellets
s.t	$4\,p_1 + 3\,p_2 + 4\,p_3 \leq 5$	Spaceburgers
	$2\,p_1 + 4\,p_2 + 1\,p_3 \leq 4$	Saturn shakes
	$2\,p_3 \leq 2$	Venus flies
	$p_1, p_2, p_3 \geq 0$	

Complementary Slackness Conditions

Let x* denote a primal feasible solution.

Let p* denote a dual feasible solution.

Complementary slackness conditions.

1. **For all i, if the i-th constraint is an inequality constraint that is not tight at the optimum solution, then $p^*_i = 0$.**
2. **If the $x^*_j > 0$, then the j-th constraint of the dual problem is tight, and thus the reduced cost of x^*_j is 0.**

Theorem. ***If x* and p* are optimal primal and dual solutions, then they satisfy the complementary slackness conditions.***

Complementary Slackness Conditions

$$\begin{array}{llcl}
\min & v = 5x_1 + 4x_2 + 2x_3 & & \\
\text{s.t} & 4x_1 + 2x_2 & \geq 12 & \text{carbs} \\
 & 3x_1 + 4x_2 & \geq 8 & \text{fat} \\
 & 4x_1 + 1x_2 + 2x_3 & \geq 20 & \text{protein} \\
 & x_1, x_2, x_3 \geq 0 & &
\end{array}$$

Opt Solution: $x_1 = 3$, $x_2 = 0$, $x_3 = 4$, $z = 23$

What does complementary slackness tell us about the optimal dual variables (also known as shadow prices? Discuss it with your neighbor.

$p_2 = 0$; the reduced costs of x_1 and x_3 are 0.

$p_2 = 0$; $4p_1 + 3p_2 + 4p_3 = 5$; $2p_3 = 2$.

More on complementary slackness

- **If there is a unique optimum tableau, then the complementary slackness conditions will give the equations for a unique optimum dual solution.**

Dual Linear Program

$$\begin{aligned} \text{Max} \quad & z = 12\,p_1 + 8\,p_2 + 20\,p_3 \\ \text{s.t} \quad & 4\,p_1 + 3\,p_2 + 4\,p_3 \le 5 \\ & 2\,p_1 + 4\,p_2 + 1\,p_3 \le 4 \\ & 2\,p_3 \le 2 \\ & p_1, p_2, p_3 \ge 0 \end{aligned}$$

$p_2 = 0;$
$4\,p_1 + 3\,p_2 + 4\,p_3 = 5;$
$2\,p_3 = 2.$

$p_2 = 0$
$p_3 = 1$
$p_1 = 1/4$

Theorem on Complementary Slackness

If x^* is feasible for the primal, and if p^* is feasible for the dual, and if the pair (x^*, p^*) satisfy complementary slackness conditions, then

1. **x^* is optimal for the primal problem and**
2. **p^* is optimal for the dual problem**

Primal solution:
$x_1 = 3$
$x_2 = 0$
$x_3 = 4$

Dual Solution
$p_1 = 1/4$
$p_2 = 0$
$p_3 = 1$

The complementary slackness conditions are satisfied. Thus they are both optimal.

Complementary Slackness: An alternative representation

Primal solution:
$x_1 = 3$
$x_2 = 0$
$x_3 = 4$

Dual Solution
$p_1 = 1/4$
$p_2 = 0$
$p_3 = 1$

$x_1 \times (4\ p_1 + 3\ p_2 + 4\ p_3 - 5) = 0$ $\qquad 3 \times 0 = 0$

$x_2 \times (2\ p_1 + 4\ p_2 + 1\ p_3 - 4) = 0$ $\qquad 0 \times -2.5 = 0$

$x_3\ (2\ p_3 - 2) = 0$ $\qquad 4 \times 0 = 0$

Mental Break

The Dual Simplex Algorithm

z	x_1	x_2	x_3	x_4			
1	3	-2	0	0	=	0	Objective function.
0	-3	3	1	0	=	6	Constraint 1
0	-4	2	0	1	=	2	Constraint 2

If there are m constraints, a ***basic solution*** has m basic variables.

The basic solution is obtained by setting the nonbasic variables to 0, and solving the system of equations.

Basic Feasible Solutions

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

A solution is called a ***basic feasible solution***, (bfs) if it is a basic solution and if it satisfies the nonnegativity constraints.

z	x_1	x_2	x_3	x_4		
1	-1	0	0	1	=	2
0	3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

Another bfs

Basic Dual Feasible Bases

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

A solution is called a ***basic dual feasible solution***, if it is a basic solution and if it satisfies the optimality conditions with the z-row coefficients ≥ 0 (reduced costs are non-negative.)

z	x_1	x_2	x_3	x_4		
1	1	0	2/3	0	=	4
0	-1	1	1/3	0	=	2
0	-2	0	-2/3	1	=	-2

This tableau gives a ***basic dual feasible solution***. The primal basic solution is $x_2 = 2$ and $x_4 = -2$, and so it is not a bfs.

How did you know where to pivot to get that dual feasible basis?

Where are the dual variables?

Why are we learning this?

There is no general rule on where to pivot. Just focus on the fact that it shows it is a basic dual solution.

The dual variables are "hidden". They are the simplex multipliers.

We are learning this to learn the dual simplex algorithm.

Ollie,
the computationally
wise owl.

Tim

Pivoting in the Dual Simplex Algorithm

z	x_1	x_2	x_3	s_1	s_2	s_3		RHS
1	-52	-30	-20	0	0	0	=	0
0	2	4	5	1	0	0	=	100
0	1	1	1	0	1	0	=	30
0	10	5	2	0	0	1	=	204

Zor's initial tableau

z	x_1	x_2	x_3	s_1	s_2	s_3		RHS	Prices
1	48	20	0	0	0	10	=	2040	
0	-23	-8.5	0	1	0	-2.5	=	-410	0
0	-4	-1.5	0	0	1	-0.5	=	-72	0
0	5	2.5	1	0	0	0.5	=	102	10

A dual feasible tableau

A Dual Feasible Pivot

z	x_1	x_2	x_3	s_1	s_2	s_3		RHS
1	48	20	0	0	0	10	=	2040
0	-23	-8.5	0	1	0	-2.5	=	-410
0	-4	-1.5	0	0	1	-0.5	=	-72
0	5	2.5	1	0	0	0.5	=	102

A dual feasible tableau

1	48 - 4Δ	20- 1.5Δ	0	0	Δ	10 - .5Δ	=	2040 - 72Δ

Step 1. Choose a row that is an infeasible constraint.

Step 2. Add Δ times that row to the z-row.

Step 3. Choose Δ maximum so that all coefficients are non-negative. This determines the pivot column.

Finding the max value of Δ.

z	x_1	x_2	x_3	s_1	s_2	s_3		RHS
1	48	20	0	0	0	10	=	2040
0	-23	-8.5	0	1	0	-2.5	=	-410
0	-4	-1.5	0	0	1	-0.5	=	-72
0	5	2.5	1	0	0	0.5	=	102

1	48 - 4Δ	20- 1.5Δ	0	0	Δ	10 - .5Δ	=	2040 - 72Δ

48/4	20/1.5	10/.5

s = argmin { $\bar{c}_j$/ $\bar{a}_{2j}$: $\bar{a}_{2j}$ < 0}. {48/4, 20/1.5, 10/.5}.

s = 1. Pivot on the component in constraint 2, column x_1.

If $\bar{a}_{2j} \geq 0$, the dual is unbounded from below, and the primal is infeasible.

The Dual Simplex Pivot

z	x_1	x_2	x_3	s_1	s_2	s_3		RHS
1	0	2	0	0	12	4	=	1176
0	0	1/8	0	1	-23/4	3/8	=	4
0	1	3/8	0	0	- 1/4	1/8	=	18
0	0	5/8	1	0	5/4	- 1/8	=	12

Divide constraint 2 by -4.

Subtract 48 times the new constraint 2 from the z-row

Add 23 times the new constraint 2 to constraint 1.

Subtract 5 times the new constraint 2 from constraint 31.

The resulting tableau will be dual feasible. If it is also a bfs, then it is optimal, and we can quit. Otherwise keep pivoting.

Another dual simplex pivot

z	x_1	x_2	x_3	s_1	s_2	s_3		RHS
1	2	3	1	0	0	0	=	5
0	2	4	5	1	0	0	=	10
0	-2	-1	1	0	1	0	=	-3
0	1	1	1	0	0	1	=	-2

1. Does this tableau give an upper bound on the optimal primal objective value?
2. What happens if we choose the second constraint as the pivot row?
3. What happens if we choose the third constraint at the pivot row?

Answers

Answer 1. The optimal primal value is at most 5 because $z + 2x_1 + 3x_2 + x_3 = 5$ and $x \geq 0$.

What happens if we choose the second constraint as the pivot row?

The min ratio test gives min {2/2, 3/1} = 1

The column giving the min ratio is the column for x_1.
So, we pivot on the -2

What happens if we choose the third constraint as the pivot row?

The min ratio test gives nothing. The dual is unbounded.

The constraint is "$x_1 + x_2 + x_3 + s_1 = -2$"
The primal has no feasible solution.

Comments on the Dual Simplex Algorithm

- **Alternative to primal simplex algorithm in which we relax primal feasibility, but require the z-row to satisfy the optimality conditions.**
- **Dual simplex pivot rule: determined by a min ratio test.**
- **Very efficient in practice for lots of large problems (great alternative to primal simplex)**
- **If add an inequality constraint to an LP, the current basis may not be a bfs, but it stays dual feasible. (Very common in integer programming.)**
  - **The old basis can be used as a starting point for the dual simplex method**

And now, it's time for …..