

**15.053**

**Tuesday, April 24**

**Integer Programming Formulations 2**

## **Quote of the Day**

**What chiefly characterizes creative thinking from more mundane forms are (i) willingness to accept vaguely defined problem statements and gradually structure them, (ii) continuing preoccupation with problems over a considerable period of time, and (iii) extensive background knowledge in relevant and potentially relevant areas.**

**-- Herbert Simon**

## Overview of lecture

- **First half of lecture:**
  - more logical constraints
  - non-linearities
- **Second half of lecture**
  - classic combinatorial problems
    - set covering
    - facility location
    - graph coloring
    - exam scheduling

## Logical Constraints with non-binary variables

Variables:  $x_1, x_2, x_3$ , are non-negative integers

$$x_1 \leq 10 \quad \text{or} \quad x_2 \geq 25 \quad \text{or both}$$

Assume that  $x_1 \leq 100$ . We need an upper bound.

Create a variable  $w_1$  with the following properties:

If  $w_1 = 1$ , then  $x_1 \leq 10$ ;

If  $w_1 = 0$ , then  $x_2 \geq 25$ ;  $w_1 = 0$  or  $1$ .

$$x_1 \leq 9 + 91(1-w_1)$$

$$x_2 \geq 25 - 25w_1$$

$w_1 = 1$	$w_1 = 0$
$x_1 \leq 9$	$x_1 \leq 100$
$x_2 \geq 0$	$x_2 \geq 25$

## “Or-constraints” in general.

$$ax \leq b \quad \text{or} \quad cx \geq d \quad \text{or both}$$

Let  $M_1$  and  $M_2$  be integers so that  
 $ax \leq M_1$  and  $cx \geq M_2$  in any feasible solution.

Typically  $M_1$  is very large, and  $M_2$  is “very negative”

If  $w_2 = 1$ , then  $ax \leq b$ ;

If  $w_2 = 0$ , then  $cx \geq d$ ;  $w_2 = 0$  or  $1$ .

$$ax \leq b + (M_1 - b)(1-w_2)$$
$$cx \geq d + (M_2 - d)w_2$$

$w_2 = 1$	$w_2 = 0$
$ax \leq b$	$ax \leq M_1$
$cx \geq M_2$	$cx \geq d$

## Another example

$$|x_1 - x_2| \geq 10$$

Assume  $x_1 \leq 100$  and  $x_2 \leq 200$ .

$$\begin{aligned} \text{Translation: } x_1 - x_2 &\geq 10 \\ \text{or } x_2 - x_1 &\geq 10 \end{aligned}$$

$$\begin{aligned} \text{Note: } x_1 - x_2 &\geq -200 \\ x_2 - x_1 &\geq -100 \end{aligned}$$

If  $w_3 = 1$ , then

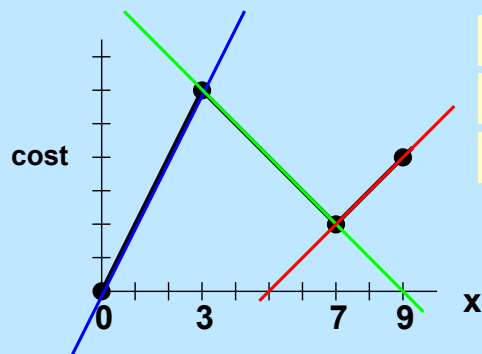
If  $w_3 = 0$ , then

$$\begin{aligned} x_1 - x_2 &\geq 10 + \\ x_2 - x_1 &\geq \end{aligned}$$

Fill in the table

$w_3 = 1$	$w_3 = 0$
$x_1 - x_2 \geq$	$x_1 - x_2 \geq$
$x_2 - x_1 \geq$	$x_2 - x_1 \geq$

## IP and piecewise linear functions.



$$f(x) = 2x \quad \text{if } 0 \leq x \leq 3$$

$$f(x) = 9 - x \quad \text{if } 4 \leq x \leq 7$$

$$f(x) = -5 + x \quad \text{if } 8 \leq x \leq 9$$

Assume that  $x$  is integer valued.

An NLP formulation

How do we model the function  $f(x)$ ?

Later we will assume that  $f(x)$  is part of the objective function, and we will maximize it.

## Version 1: Create Binary Variables

Step 1. Create three binary variables,  $w_1$ ,  $w_2$ , and  $w_3$

$$w_1 + w_2 + w_3 = 1$$

If  $0 \leq x \leq 3$  then  $w_1 = 1$

If  $4 \leq x \leq 7$  then  $w_2 = 1$

If  $8 \leq x \leq 9$  then  $w_3 = 1$

$w_j \in \{0, 1\}$  for  $j = 1, 2, 3$ .

Almost works, but we need something else.

Warning, this transformation will be challenging to follow the first time around.

## Version 2: Split $x$ into three variables

Step 2. Split  $x$  into three variables  $x_1, x_2, x_3$ .

**If  $0 \leq x \leq 3$  then  $x_1 = x$ ; otherwise,  $x_1 = 0$ .**

**If  $4 \leq x \leq 7$  then  $x_2 = x$ ; otherwise,  $x_2 = 0$ .**

**If  $8 \leq x \leq 9$  then  $x_3 = x$ ; otherwise,  $x_3 = 0$ .**

$$x = x_1 + x_2 + x_3.$$

## Version 2 continued. Combine Steps 1 and 2.

### Transformation

$$w_1 + w_2 + w_3 = 1$$

$$0 \leq x_1 \leq 3 w_1$$

$$4w_2 \leq x_2 \leq 7 w_2$$

$$8w_3 \leq x_3 \leq 9 w_3$$

$$x = x_1 + x_2 + x_3$$

$$f(x) = 2x_1 + (9w_2 - x_2) + (-5w_3 + x_3)$$

### Last step

$$z = 2x \quad \text{if } 0 \leq x \leq 3$$

$$z = 9 - x \quad \text{if } 4 \leq x \leq 7$$

$$z = -5 + x \quad \text{if } 8 \leq x \leq 9$$

10

$$w_1 + w_2 + w_3 = 1$$

If  $w_1 = 1$ , then  $0 \leq x_1 \leq 3$

If  $w_1 = 0$ , then  $x_1 = 0$

If  $w_2 = 1$ , then  $4 \leq x_2 \leq 7$

If  $w_2 = 0$ , then  $x_2 = 0$

If  $w_3 = 1$ , then  $8 \leq x_3 \leq 9$

If  $w_3 = 0$ , then  $x_3 = 0$

$$x = x_1 + x_2 + x_3$$

## Another non-linear function

$$z = 2x_1 + 5x_2 + f(x_3)$$

$$f(x_3) = \begin{cases} 1 & \text{if } 0 \leq x_3 \leq 2 \\ 4 & \text{if } 3 \leq x_3 \leq 6 \\ x_3 & \text{if } 7 \leq x_3 \leq 10 \end{cases}$$

$$w_1 + w_2 + w_3 = 1$$

$$x_3 = x_{31} + x_{32} + x_{33}$$

$$\text{If } w_1 = 1, \text{ then } 0 \leq x_{31} \leq 2$$

$$\text{If } w_1 = 0, \text{ then } x_{31} = 0$$

$$\text{If } w_2 = 1, \text{ then } \boxed{\phantom{0 \leq x_{32} \leq 2}}$$

$$\text{If } w_2 = 0, \text{ then } x_{32} = 0$$

$$\text{If } w_3 = 1, \text{ then } \boxed{\phantom{0 \leq x_{33} \leq 10}}$$

$$\text{If } w_3 = 0, \text{ then } x_{33} = 0$$

$$w_1 + w_2 + w_3 = 1$$

$$x_3 = x_{31} + x_{32} + x_{33}$$

$$0 \leq x_{31} \leq 2w_1$$

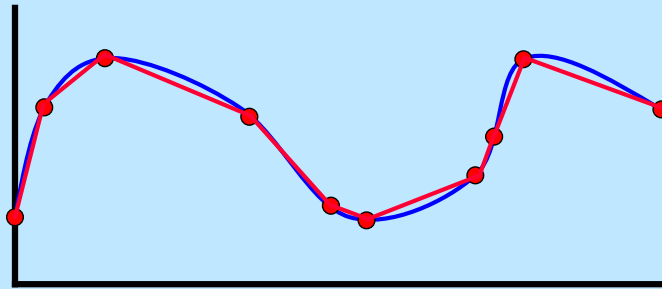
$$\boxed{0 \leq x_{32} \leq 2w_2}$$

$$\boxed{0 \leq x_{33} \leq 10w_3}$$

$$z = 2x_1 + 5x_2 +$$

## On representing non-linear functions

- Integer programs can represent piecewise linear functions of one variable. They cannot represent non-linear functions with “curves.”
- But they can nearly model non-linear functions with curves. One can create a piecewise linear function that is close to the original function.



## Mental Break

- MIT Jeopardy



## **053 Chocolates**

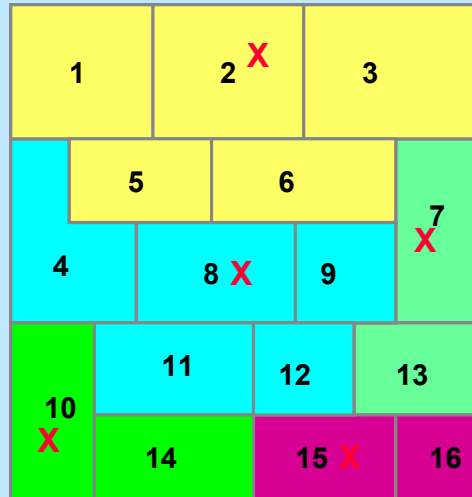
**Cleaver, Nooz, Ollie, and Tim have started a firm called 053 chocolates.**

**Slogan: “053 chocolates are optimum.”**

**Cleaver wants to ensure that everyone has easy access to the chocolates, and so requests that every district has an 053's or is next to a district with an 053's.**

**Nooz wants to minimize the number of 053 stores to satisfy the “covering constraints.” How can we satisfy the both Nooz and Cleaver?**

## 053 Chocolates



Locate 053  
Chocolate stores so  
that each district  
has a store in it or  
next to it.

Minimize the  
number of stores  
needed.

## How do we represent this as an IP?

$x_j = 1$  if set  $j$  is selected  
 $x_j = 0$  otherwise

Variables

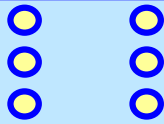
Minimize  $x_1 + x_2 + \dots + x_{16}$

Objective

s.t.  $x_1 + x_2 + x_4 + x_5 \geq 1$

$x_1 + x_2 + x_3 + x_5 + x_6 \geq 1$

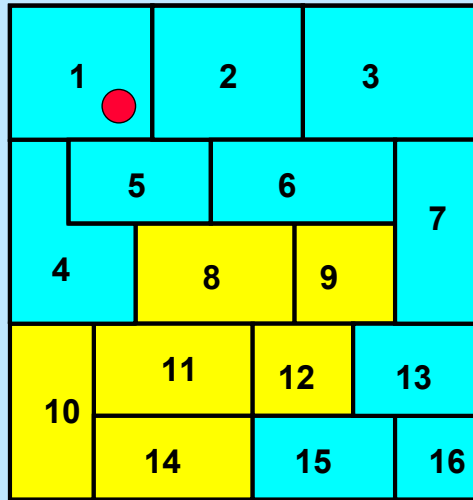
Constraints



$x_{13} + x_{15} + x_{16} \geq 1$

$x_j \in \{0, 1\}$  for each  $j$ .

## Representation as Set Covering Problem



#	Subset
1	{1, 2, 4, 5}
2	{1, 2, 3, 5, 6}
3	{2, 3, 6, 7}
	○ ○
	○ ○
	○ ○
16	{13, 15, 16}

## Set covering Problem

Let  $S = \{1, 2, \dots, n\}$ , and suppose  $S_j \subseteq S$  for each  $j$ .

We say that index set  $J$  is a cover of  $S$  if  $\bigcup_{j \in J} S_j = S$

Set covering problem: find a minimum cardinality set cover of  $S$ .

Application to locating fire stations.

Locating hospitals.

Locating Starbucks

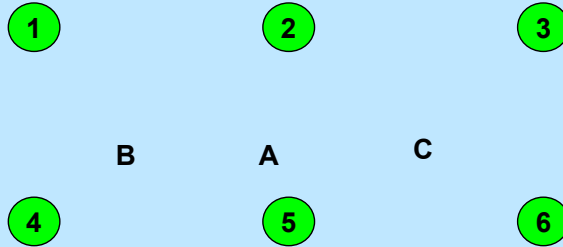
and many non-obvious applications.

## **More Integer Programming Formulations**

- **Facility Location Problems**
- **Graph Coloring**
- **Exam Scheduling**

## 053 Chocolates Location

- 6 customers.
- 3 possible locations for chocolate stores



**What is the best location for two chocolate stores so as to minimize total distance to customers?**

	1	2	3	4	5	6
A	4	1	4	4	1	4
B	2	2	6	2	2	6
C	6	2	2	6	2	2

**Distances from possible  
chocolate stores to customers**

## Formulation as an IP

Variables

$$y_j = \begin{cases} 1 & \text{if store } j \text{ is opened} \\ 0 & \text{otherwise} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if customer } i \text{ is served by store } j \\ 0 & \text{otherwise} \end{cases}$$

Objective  $\sum_i \sum_j d_{ij} x_{ij} = 4 x_{1A} + 2 x_{1B} + 6 x_{1C} + x_{2A} + \dots$

Constraints

$$x_{iA} + x_{iB} + x_{iC} = 1 \quad \text{for } i = 1 \text{ to } 6$$

$$x_{ij} \leq y_j \quad \text{for } i = 1 \text{ to } 6, \text{ and } j = A, B, C$$

$$y_A + y_B + y_C = 2$$

$$x_{ij} \in \{0,1\} \text{ for all } i, j; \quad y_j \in \{0,1\} \text{ for all } j$$

## Warehouse Location Problem

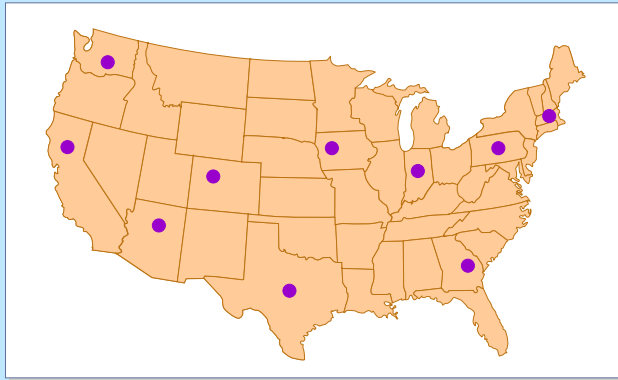


Figure by MIT OCW.

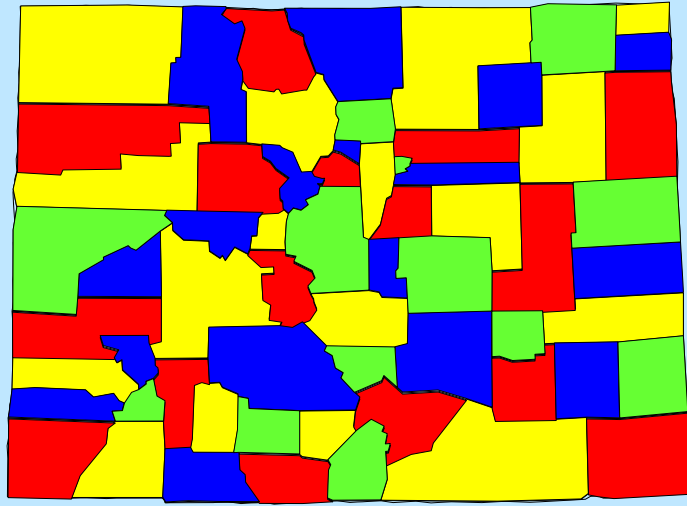
**Open warehouses so as to minimize the total cost of doing business**

## Location problems

- **Locations:** (Warehouses, or stores, or sites, or internet routers or ... ) need to be opened
- **Customers need to be served**
- **A customer cannot be served by a location unless the location is opened**
  
- **Variants:**
  - there may be a cost for opening a location
  - customers in some problems may be served by more than one location
  - There may be a capacity on how much is stored at the warehouse.

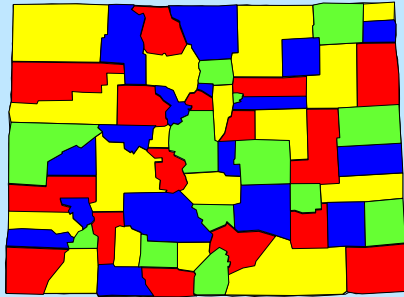


## Graph Coloring



Here is a four coloring of the map.

## Graph Coloring



$G = (N, A)$

$N = \{1, 2, 3, \dots, n\}$   
set of counties

$A =$  set of arcs.  
 $(i, j) \in A$  if counties  
 $i$  and  $j$  are adjacent.

**Exercise:** write an integer program whose solution gives the minimum number of colors to color a map.

**Key choice:** what are the decision variables?

## The Integer Programming Formulation

## An Exam Scheduling Problem

The University of Waterford has to schedule 500 exams over two weeks. The objective is to schedule the exams in the fewest number of periods so that no person has to take two exams at the same time.

$G = (N, A)$

$N = \{1, 2, 3, \dots, n\}$                       set of exams

$A =$  set of arcs.

$(i, j) \in A$  if a person needs to take exam  $i$  and exam  $j$ .

How would one model this as an integer program?

## Another Exam Scheduling Problem

$G = (N, A)$

$N = \{1, 2, 3, \dots, n\}$                       set of exams

There are  $m$  exam periods.

$A =$  set of arcs.

$(i, j) \in A$  if a person needs to take exam  $i$  and exam  $j$ .

let  $c_{ij}$  = the number of persons who need to take  
exam  $i$  and exam  $j$ .

Suppose that there are exactly  $m$  exam periods.

Formulate the problem of minimizing the number of  
people in total who are taking two exams at the same time.

## The IP formulation

Let  $x_{ik} = \begin{cases} 1 & \text{if exam } i \text{ is scheduled in period } k \\ 0 & \text{otherwise} \end{cases}$

Let  $y_{ij}^k = \begin{cases} 1 & \text{if exams } i \text{ and } j \text{ are scheduled in period } k \\ 0 & \text{otherwise} \end{cases}$

Minimize \_\_\_\_\_

subject to  $x_{ik} + x_{jk} \leq y_{ij}^k + 1$  for all  $i, j$

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\_\_\_\_\_

## Summary of Integer Programming

- **Integer Programs can model an amazing number of different things.**
  - linear constraints
  - logical constraints
  - non-linearities
- **It can be challenging to model**
- **It can be challenging to solve**
- **Next lectures: how to solve IPs**