15.070
Homework Assignment 4

Given: November 2, 2005 **Due:** November 10, 2005

Problem 1 *Consider an asymmetric simple random walk* $Q(t)$ *on* $\mathbb{Z}$ *given by* $\mathbb{P}(Q(t+1) = x+1|Q(t) = x) = p$ *and* $\mathbb{P}(Q(t+1) = x-1|Q(t) = x) = 1-p$ *for some* $0 < p < 1$.

- ***A***. *Construct a function of the state* $\phi(x), x \in \mathbb{Z}$ *such that* $\phi(Q(t))$ *is a martingale.*
- ***B***. *Suppose* $Q(0) = z > 0$ *and* $p > 1/2$. *Compute the probability that the random walk never hits* 0 *in terms of* z, p.

Problem 2 *Problem 1 in Lecture 12. You may assume a special case that* $\psi_n(t) = n$ *for all* $t \in (-\frac{1}{n}, 0)$. *NOTE: in the lecture notes we assumed that* ψ_n *is continuous, but in this special case* $\psi_n = n$ *this not required. You can also assume that* $\int_0^T X(t)dt$ *exists a.s.. i.e.,* $X(t)$ *is a.s. Riemann integrable over* $[0, T]$.

HINT: The following analysis result might be useful: A bounded function $f(x)$ *is Riemann integrable over an interval* $[a, b]$ *if and only if the set of points in* $[a, b]$ *where* $f(x)$ *is discontinuous has Lesbegue measure* 0.

Problem 3 *Problem 1 in Lecture 13. NOTE: you DO NOT need Ito isometry to prove the uniqueness of the limit (You can ignore the comment "Using this .."). But you do need to prove Ito isometry in addition to proving uniqueness.*

Problem 4 *Problem 1 in Lecture 14. Assume that* $a(t)$ *is bounded. You may use the following fact from real analysis. If* a *is integrable then* $\int_0^t a(s)ds = \lim_n \sum_j a(t_j)(t_{j+1} - t_j)$ *where the limit is along any sequence of partitions* $\Pi_n : 0 = t_0 < t_1 < \cdots < t_n = t$ *such that* $\Delta(\Pi_n) \to 0$. *(In fact this is the definition of Riemann integral).*

Problem 5 *Problem 2 in Lecture 14.*

Problem 6 *Let* $X(t) = e^{ct+\alpha B(t)}$. *Find* $dX(t)$.