

15.070

### Homework Assignment 3

Given: October 12, 2005

Due: October 19, 2005

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**Problem 1** *Problem 1 from Lecture 6.*

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**Problem 2** *Problem 6.27, Resnick's chapter 6 from course packet. A stochastic process  $\{X_t\}$  is stationary if for every  $0 < t_1 < t_2 < \dots < t_k$  and  $\Delta > 0$ , the joint distribution of  $(X(t_1), \dots, X(t_k))$  is equal to the joint distribution of  $(X(t_1 + \Delta), \dots, X(t_k + \Delta))$ .*

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**Problem 3** *Problem 2 from Lecture 8.*

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**Problem 4** (a) *Consider the following sequence of partitions  $\Pi_n, n = 1, 2, \dots$  of  $[0, 1]$  given by  $t_i = \frac{i}{n}, 0 \leq i \leq n$ . Prove that quadratic variation of a standard Brownian motion almost surely converges to 1:  $\lim_n Q(\Pi_n, B) = 1$  a.s.*

(b) *Suppose now the partition is generated by drawing  $n$  independent random values  $t_k = U_k, 1 \leq k \leq n$  drawn uniformly from  $[0, 1]$  and independently from the Brownian motion. Prove that  $\lim_n Q(\Pi_n, B) = 1$  a.s. Note, almost sure is with respect to the probability space of both the Brownian motion probability and uniform sampling.*

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**Problem 5** *Suppose  $\mathbb{E}[|X|] < \infty, X \in \mathcal{F}$ . Suppose  $\mathcal{G} \subset \mathcal{F}$  is a field which is generated by a finite partition  $A_1, \dots, A_m \subset \Omega$ . That is  $\cup A_i = \Omega, A_i \cap A_j = \emptyset, i \neq j$ . Prove that  $\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2] = \inf_Z \mathbb{E}[(X - Z)^2]$ , where  $\inf$  is over all random variables  $Z \in \mathcal{G}$ . That is, conditional probability is the best approximation of  $X$  in the mean squared metric among all r.v. measurable with respect to  $\mathcal{G}$ . Incidentally, this is true for any  $\mathcal{G} \subset \mathcal{F}$ .*

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**Problem 6** *Give an example on  $\Omega = \{a, b, c\}$  in which  $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_1]|\mathcal{F}_2] \neq \mathbb{E}[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1]$ .*

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**Problem 7** *Problem 1 from Lecture 9.*