

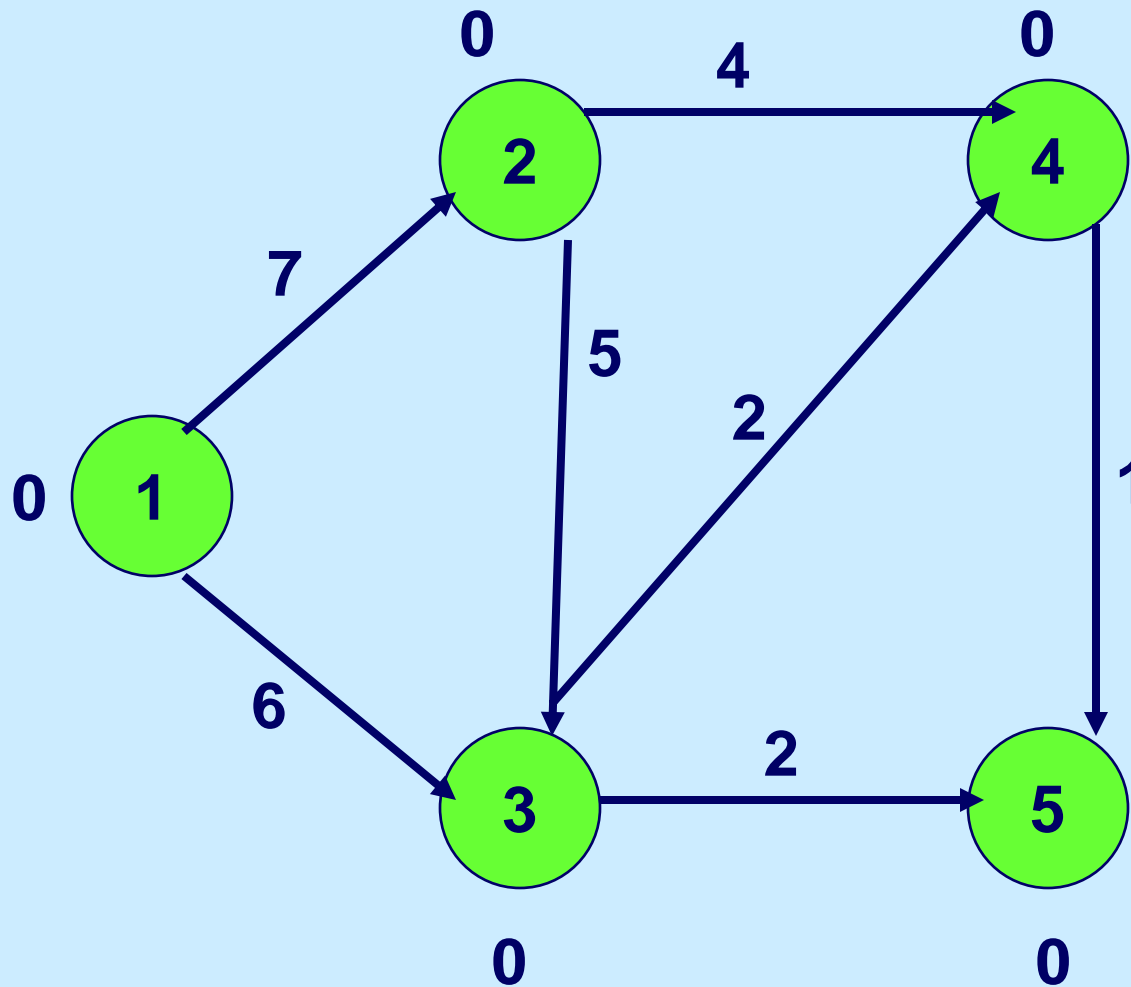
**15.082 and 6.855J**

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# **The Capacity Scaling Algorithm**

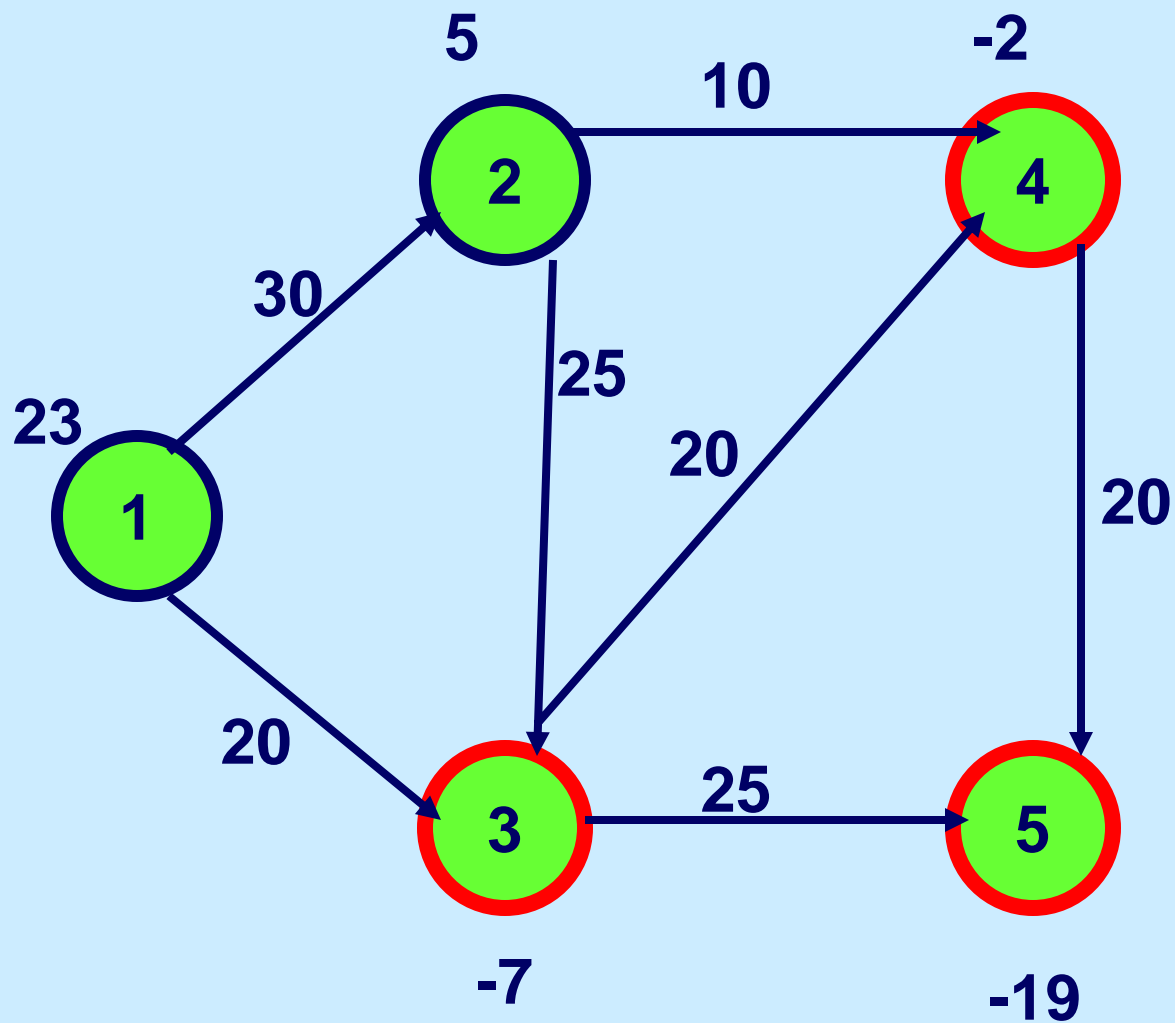
# The Original Costs and Node Potentials

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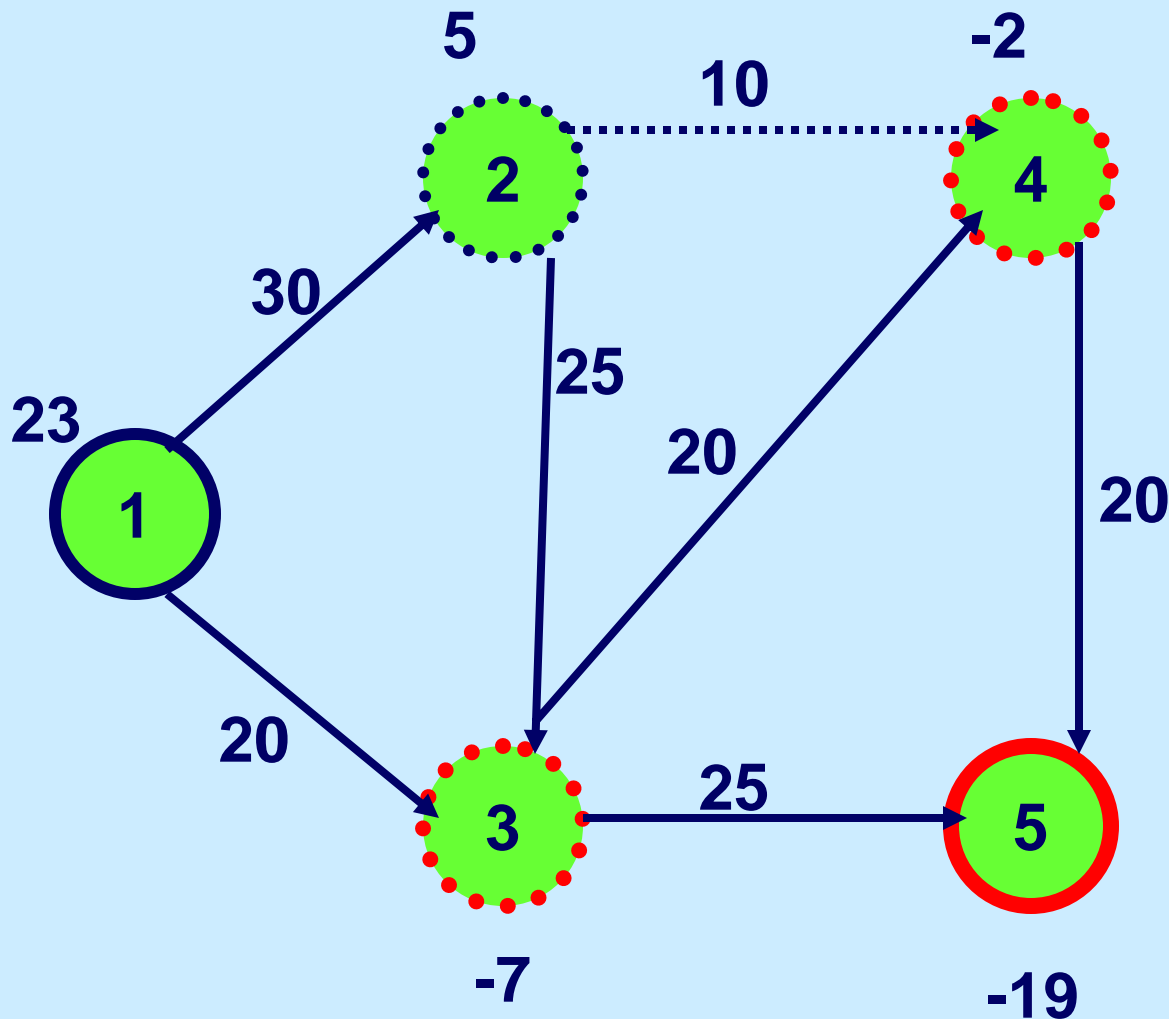
# The Original Capacities and Supplies/Demands

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Set  $\Delta = 16$ .

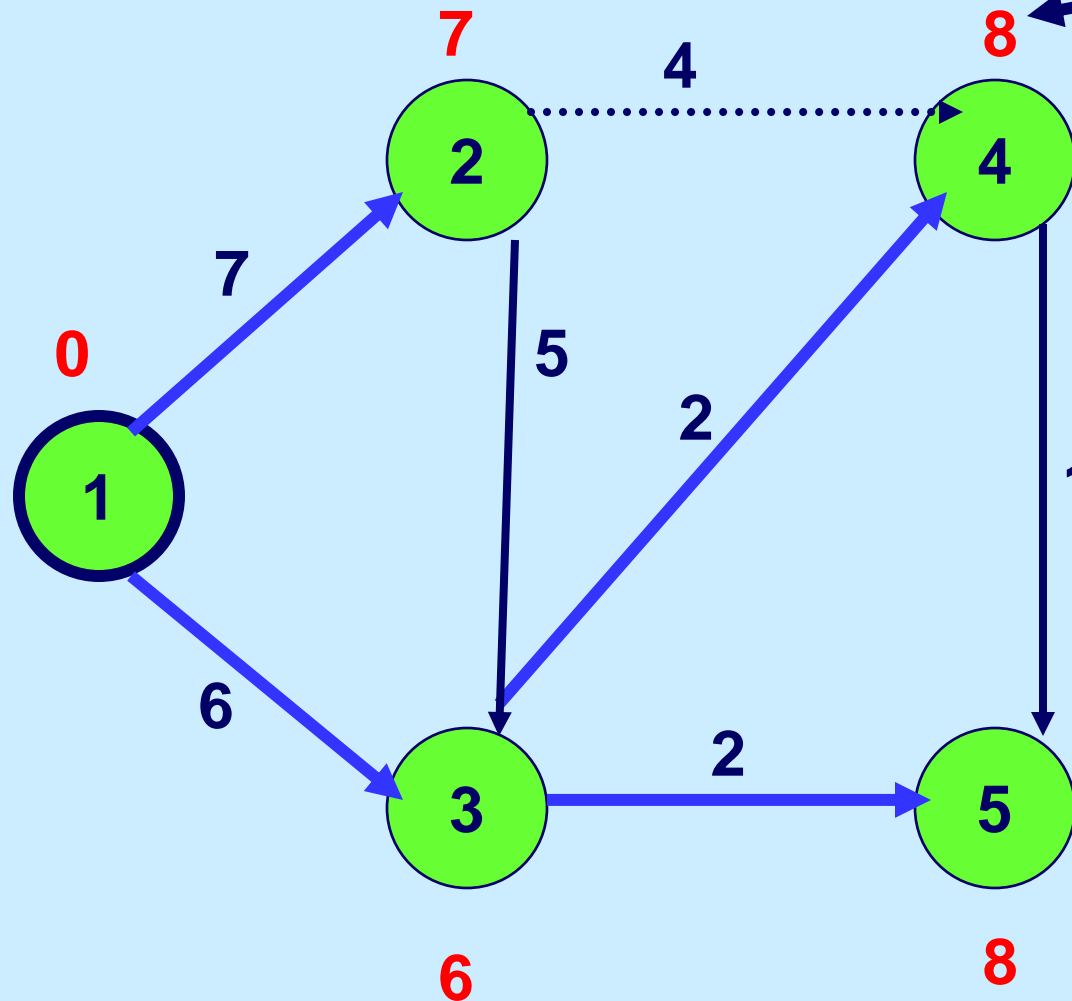
This begins the  $\Delta$ -scaling phase.



We send flow from nodes with excess  $\geq \Delta$  to nodes with deficit  $\geq \Delta$ .

We ignore arcs with capacity  $\leq \Delta$ .

# Select a supply node and find the shortest paths

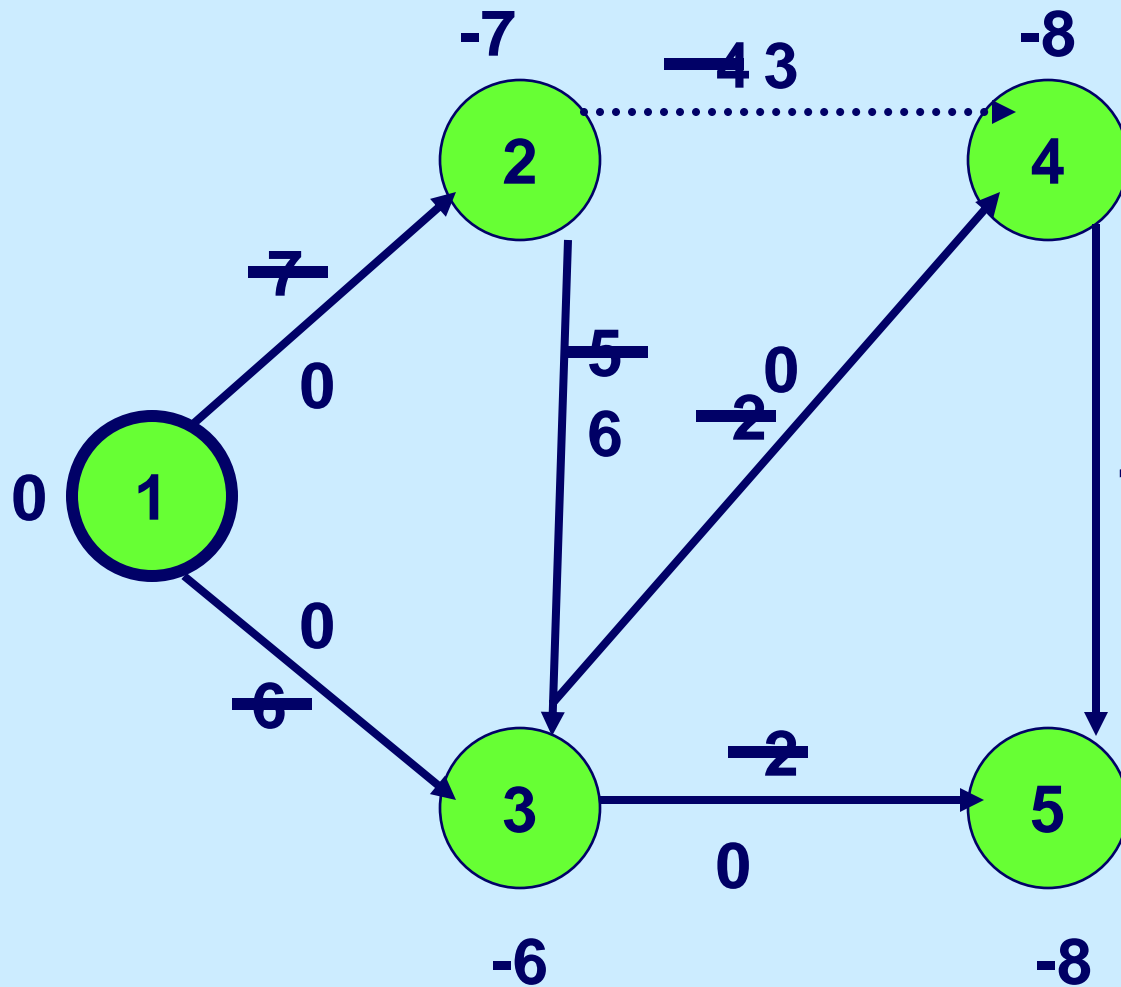


shortest path distance

The shortest path tree is marked in bold and blue.

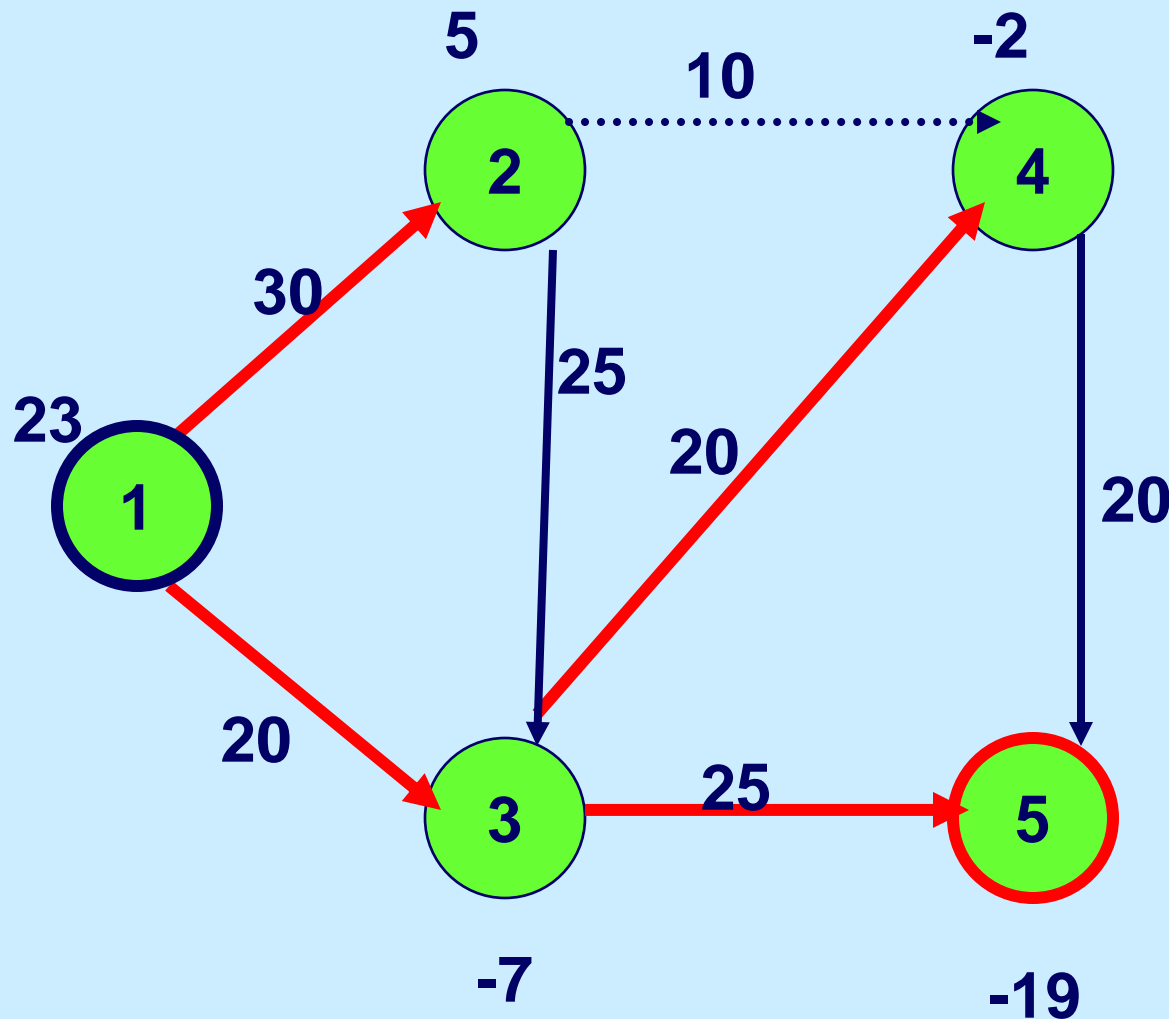
# Update the Node Potentials and the Reduced Costs

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To update a node potential, subtract the shortest path distance.

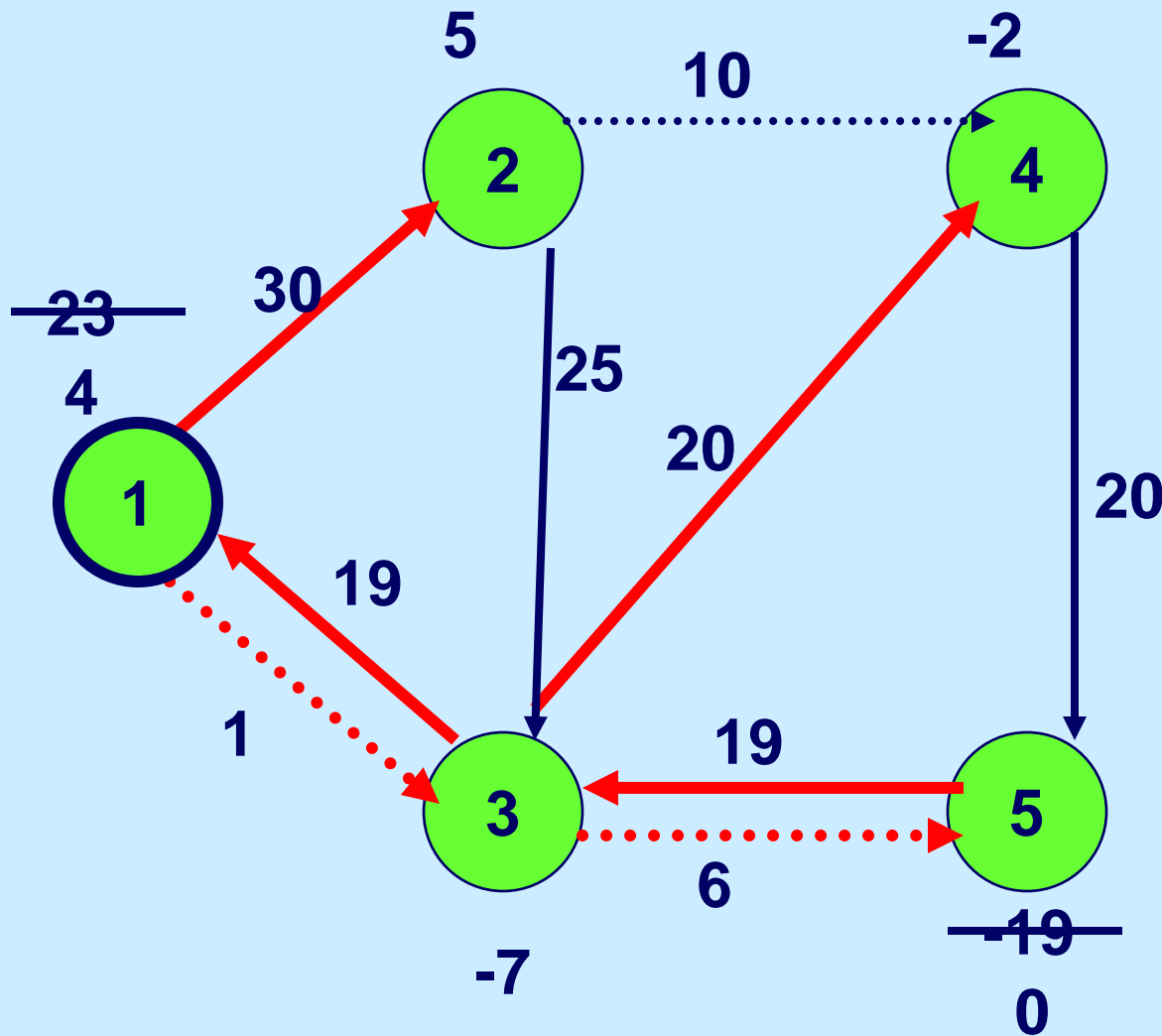
# Send Flow Along a Shortest Path in $G(x, 16)$



Send flow  
from node 1  
to node 5.

How much  
flow should  
be sent?

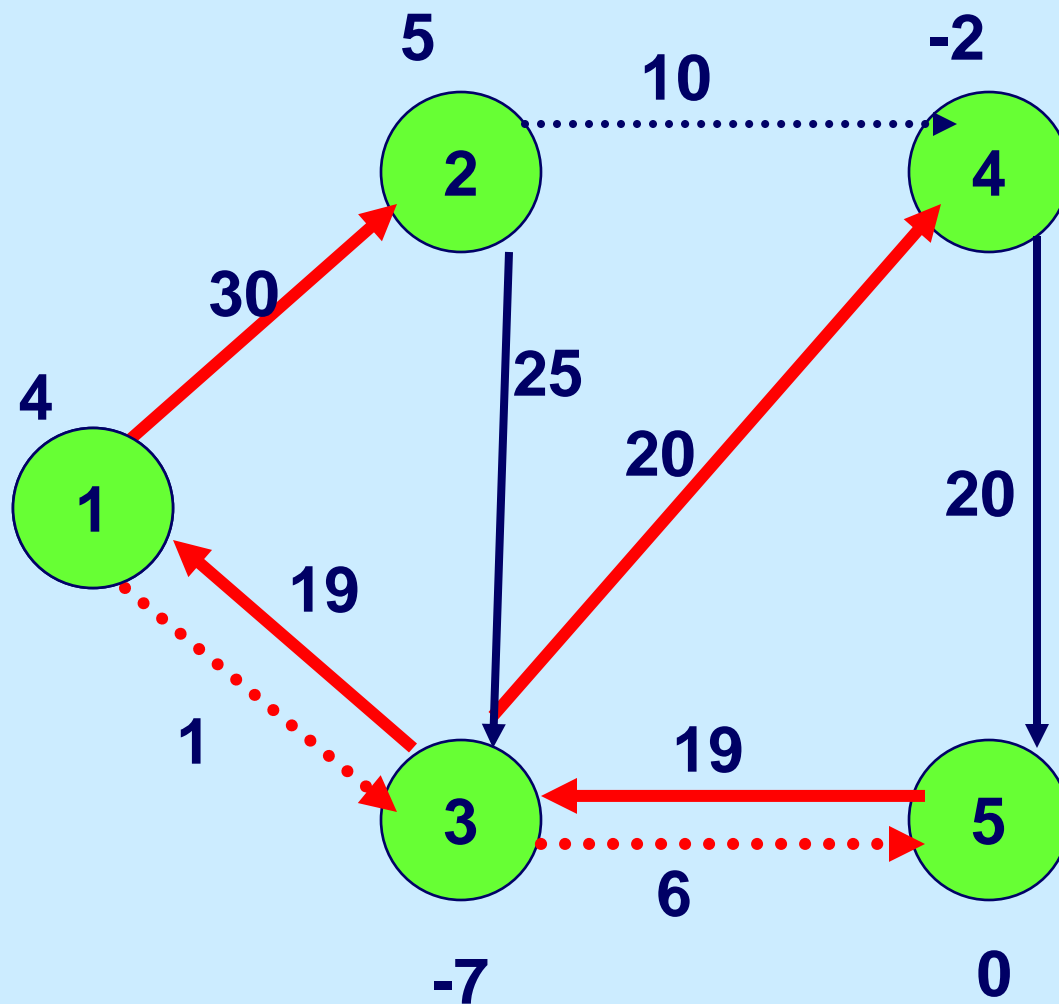
# Update the Residual Network



19 units of flow were sent from node 1 to node 5.

# This ends the 16-scaling phase.

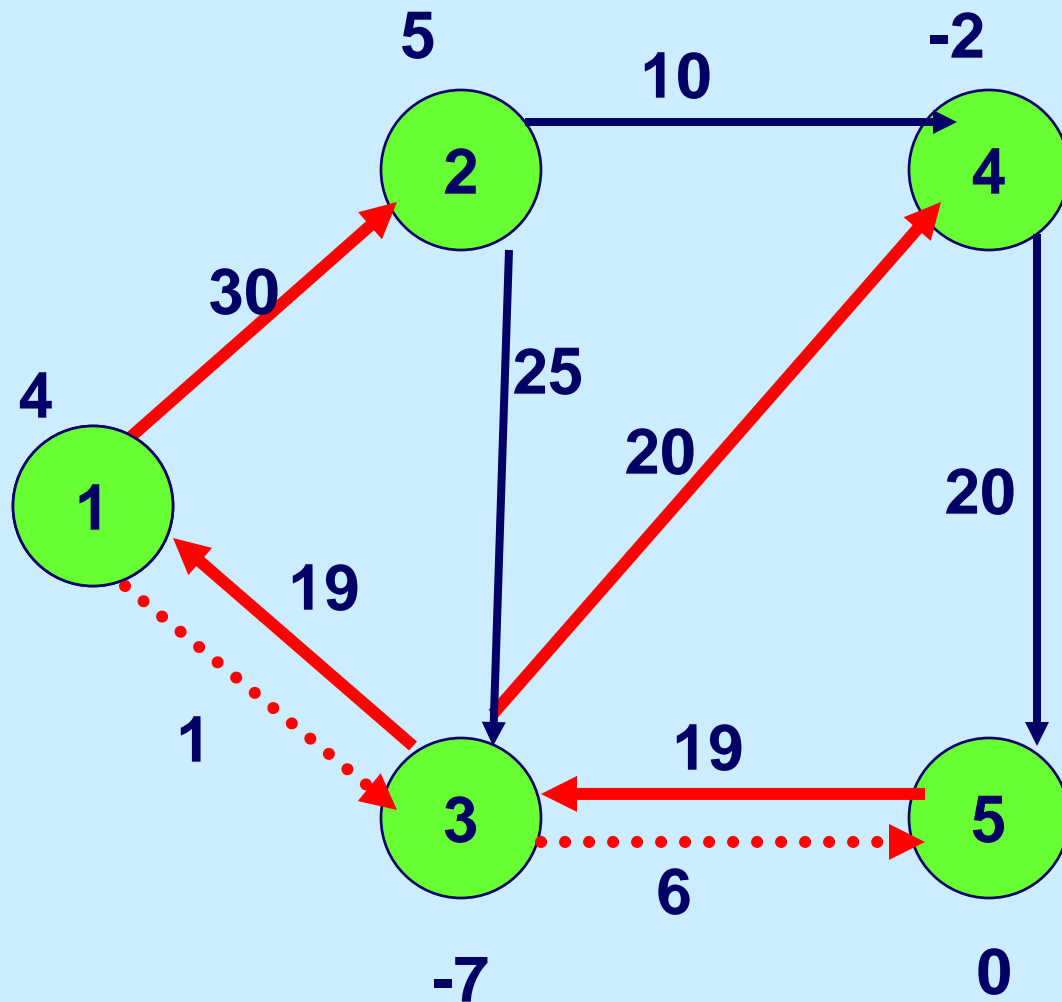
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The  $\Delta$ -scaling phase continues when  $e(i) \geq \Delta$  for some  $i$ .  
 $e(j) \leq -\Delta$  for some  $j$ .  
There is a path from  $i$  to  $j$ .

# This begins and ends the 8-scaling phase.

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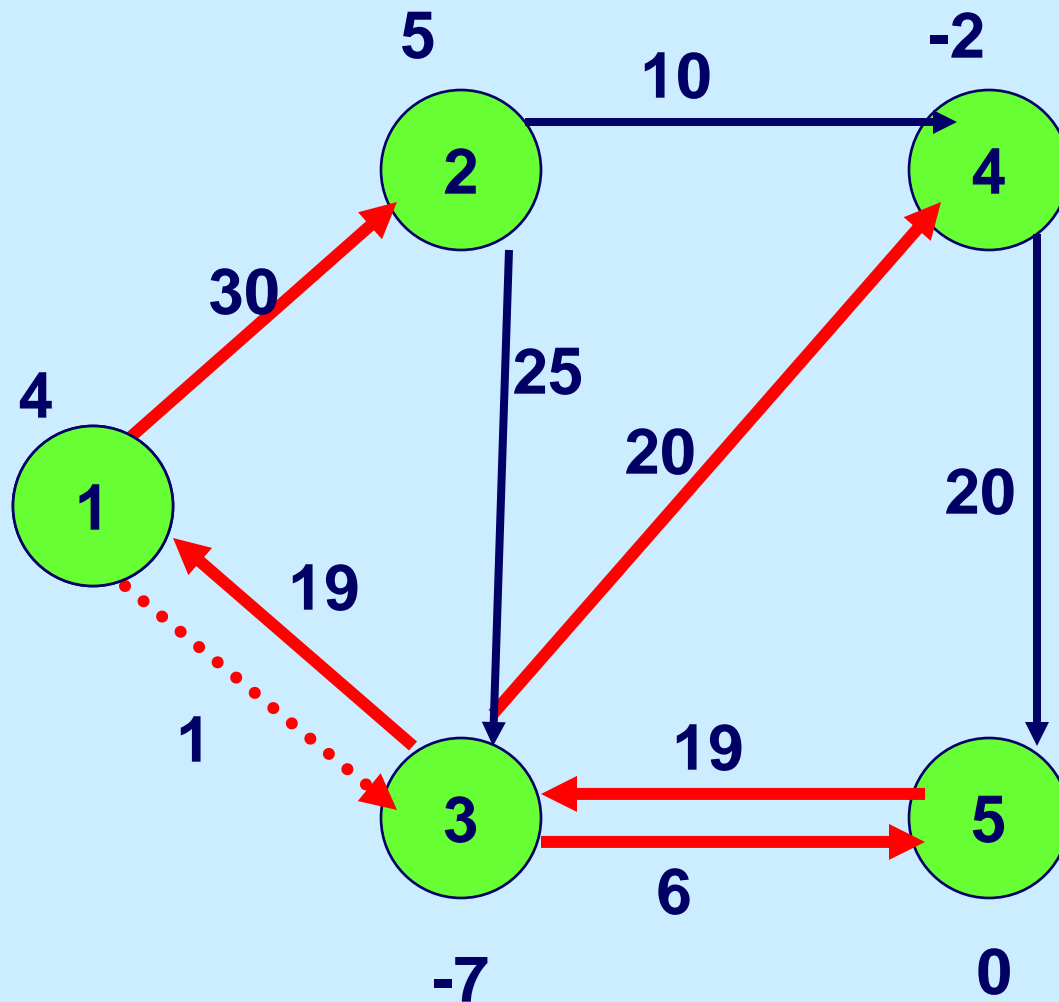
The  $\Delta$ -scaling phase continues when

- $e(i) \geq \Delta$  for some  $i$ .
- $e(j) \leq -\Delta$  for some  $j$ .

There is a path from  $i$  to  $j$ .

# This begins 4-scaling phase.

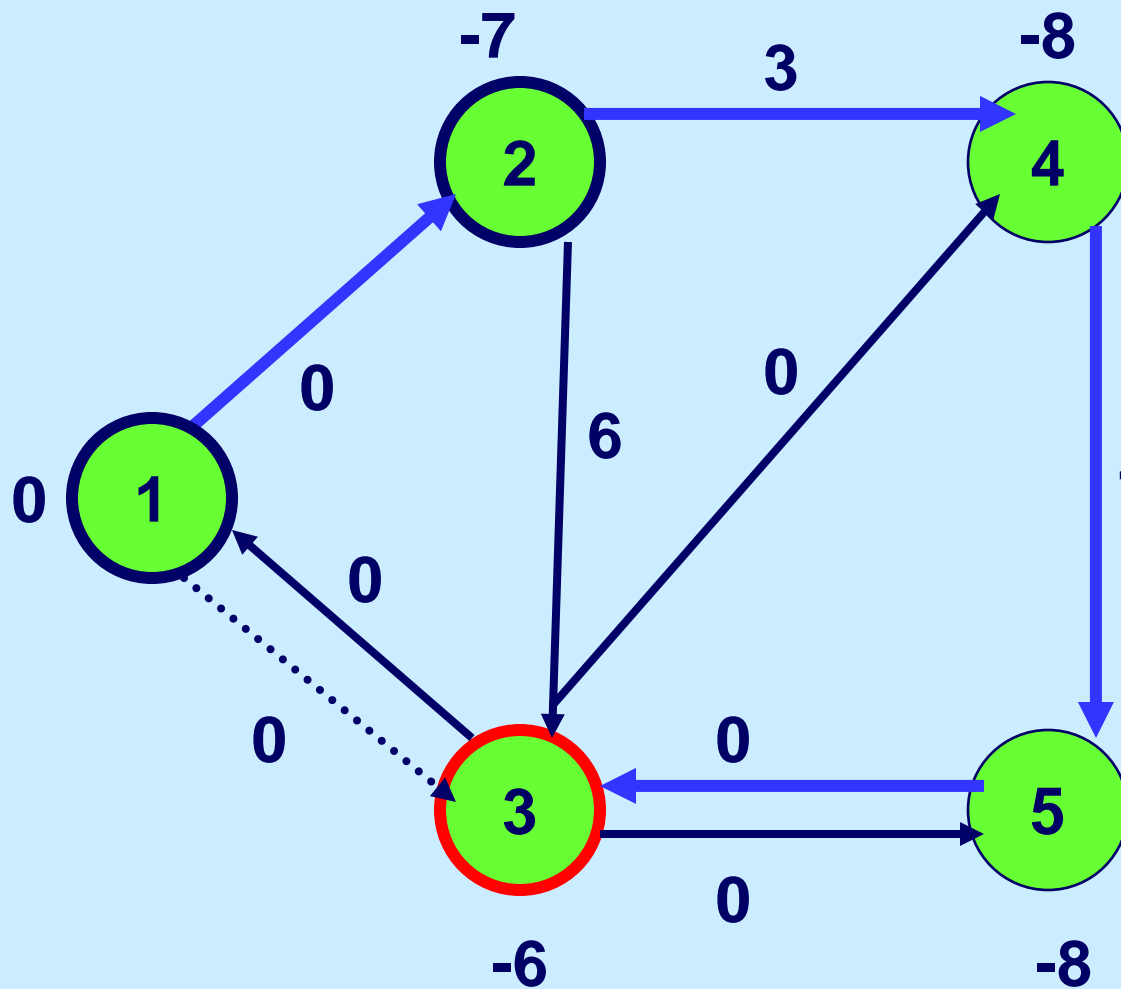
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What would we do if there were arcs with capacity at least 4 and negative reduced cost?

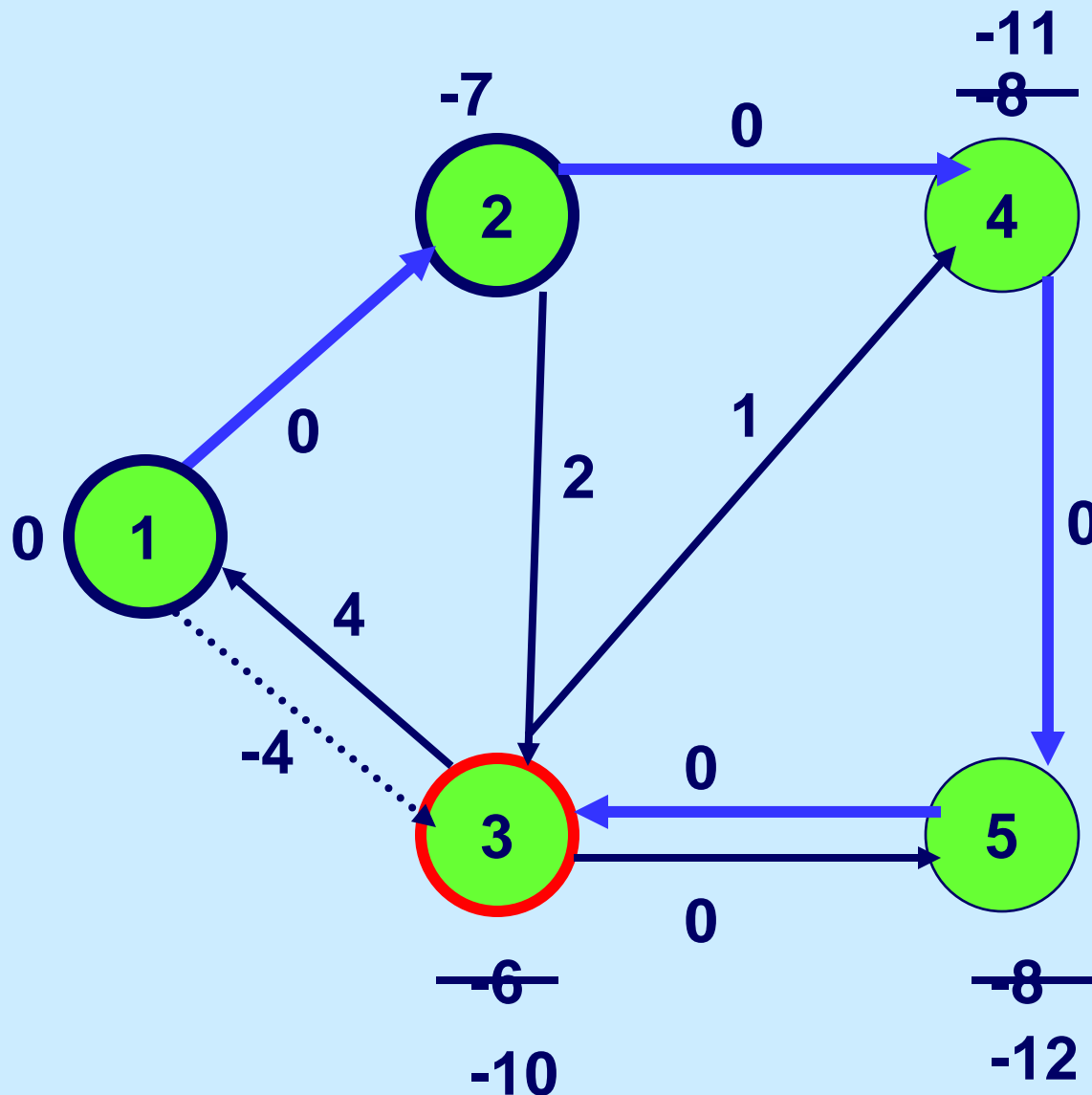
# Select a “large excess” node and find shortest paths.

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The shortest path tree is marked in bold and blue.

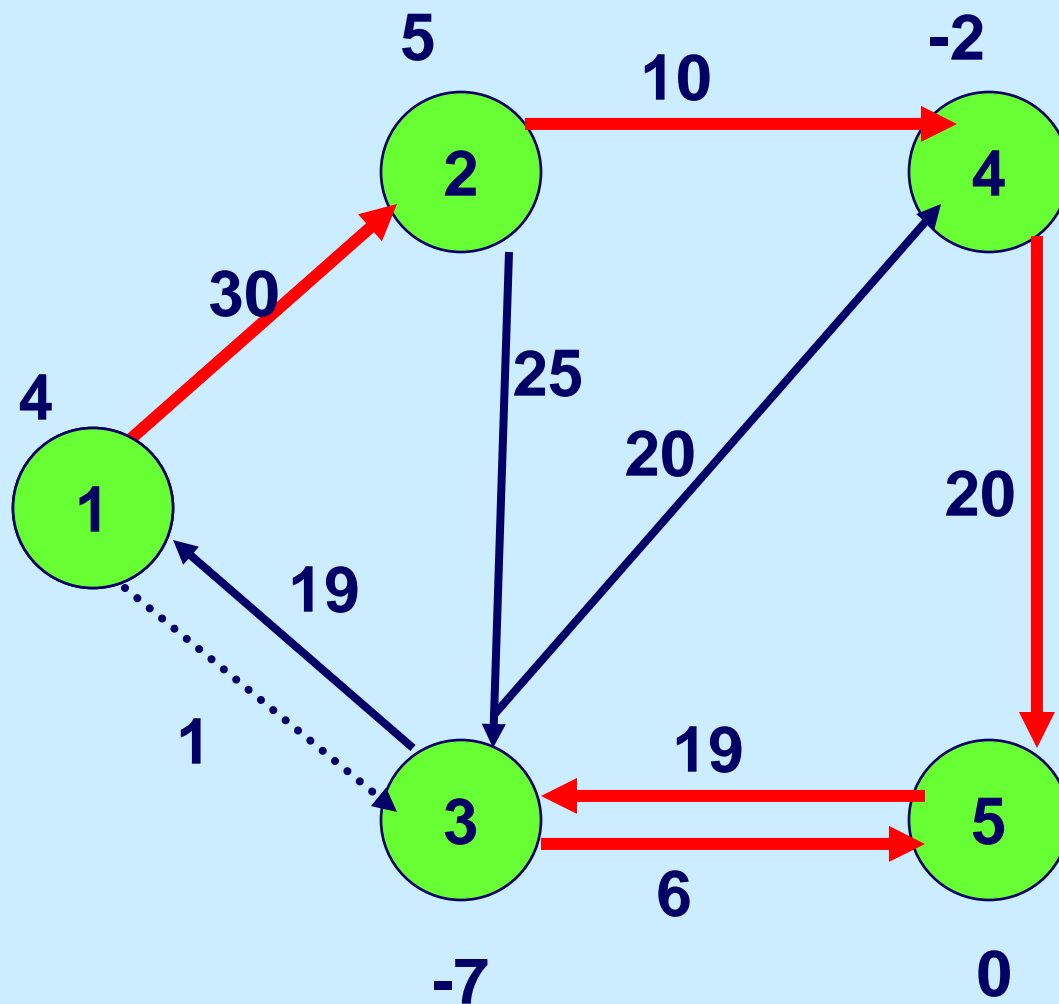
# Update the Node Potentials and the Reduced Costs



To update a node potential, subtract the shortest path distance.

**Note: low capacity arcs may have a negative reduced cost**

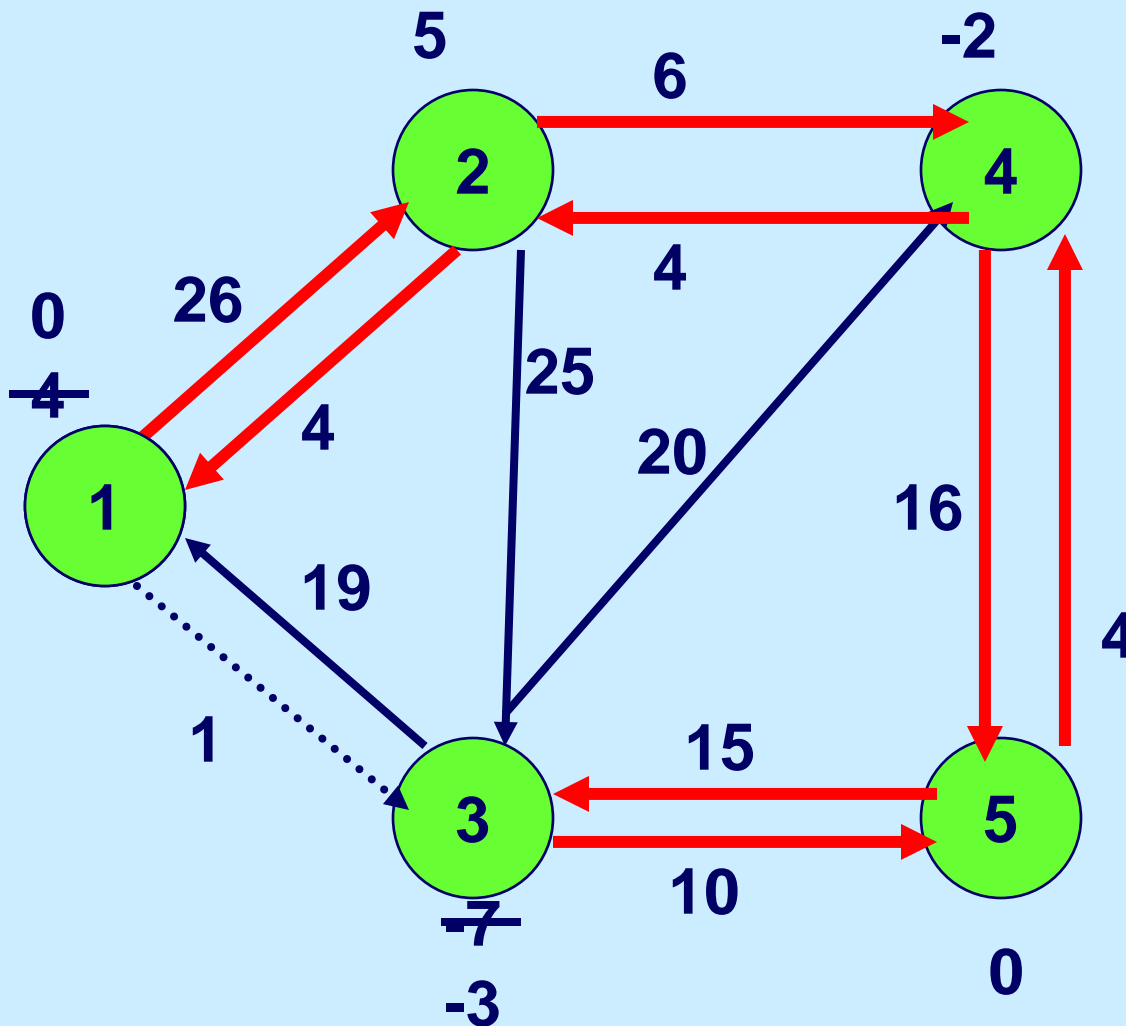
# Send Flow Along a Shortest Path in $G(x, 4)$ .



Send flow  
from node  
1 to node 7

How much  
flow should  
be sent?

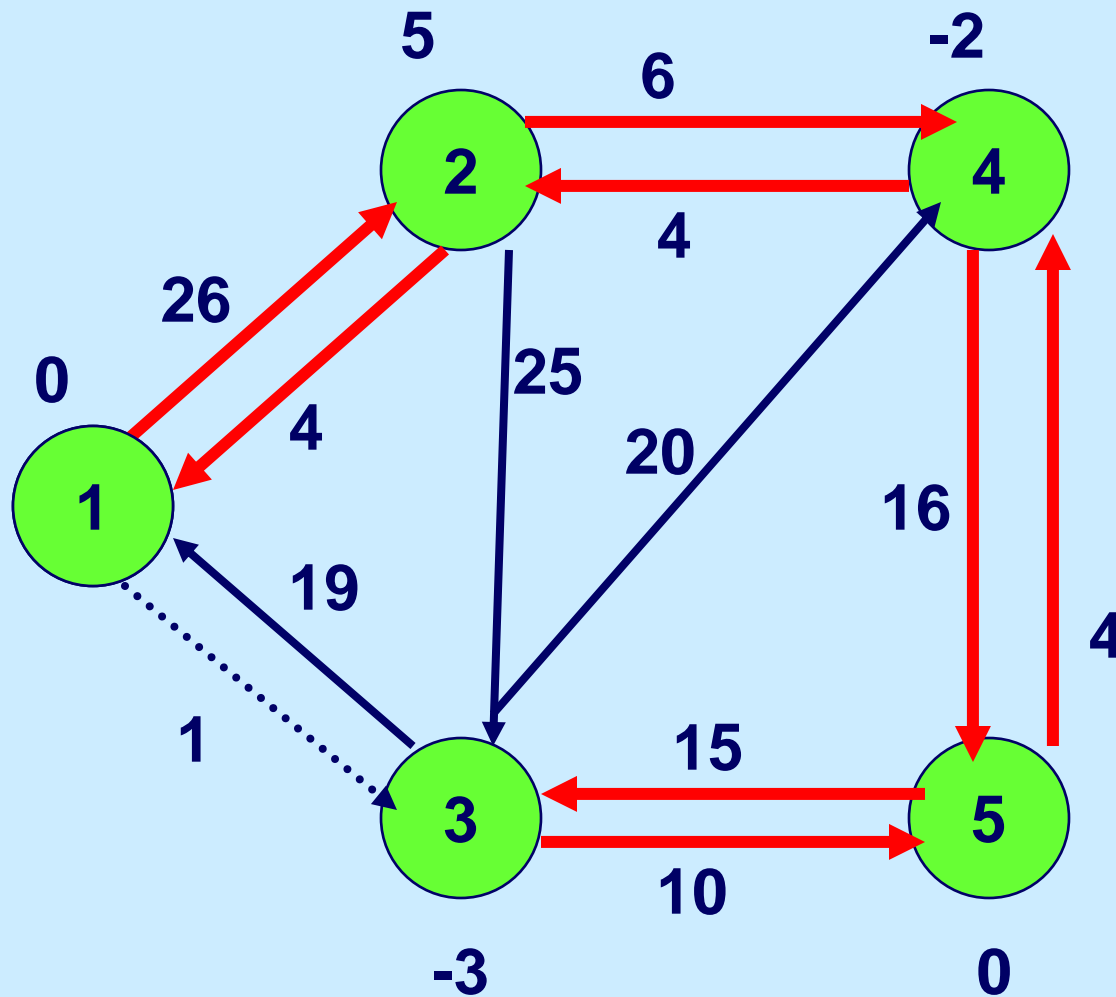
# Update the Residual Network



4 units of flow were sent from node 1 to node 3

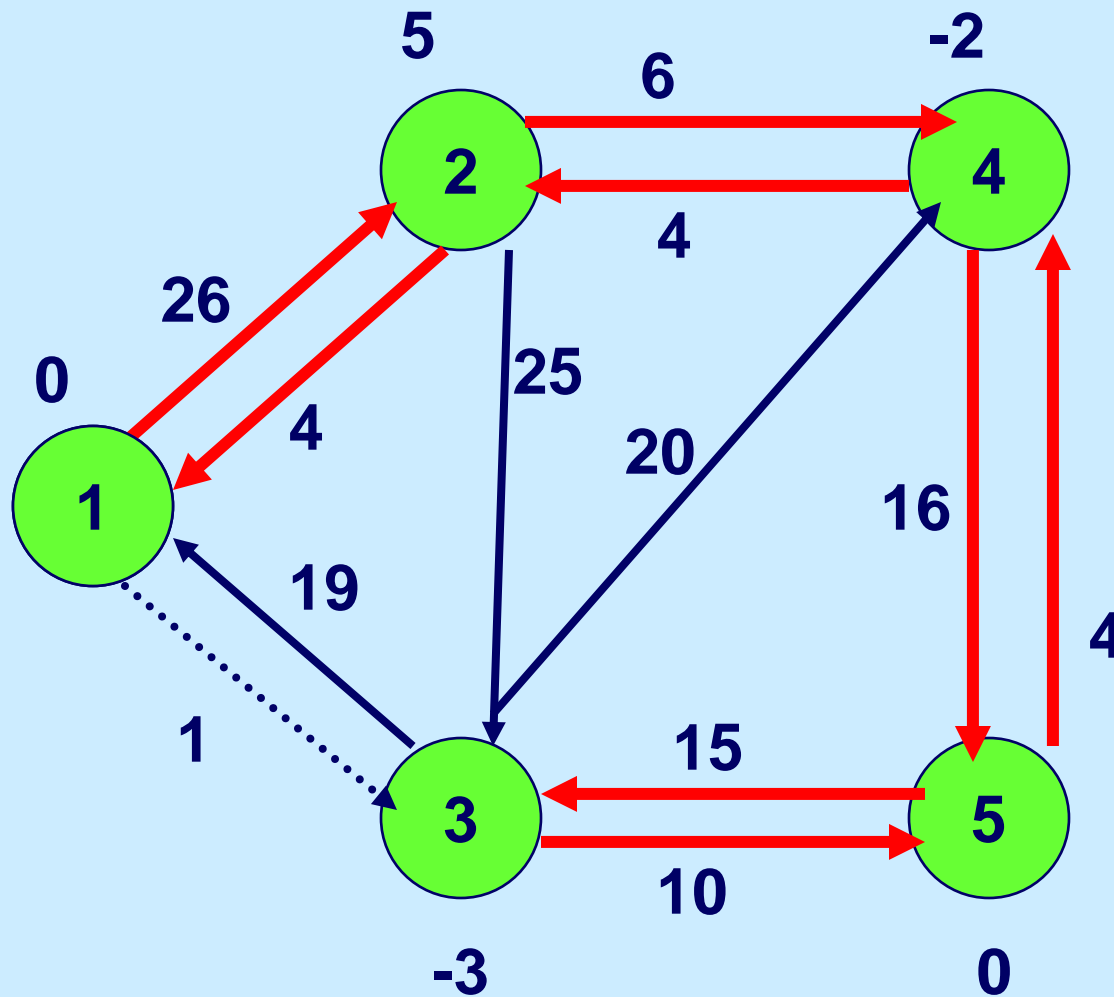
# This ends the 4-scaling phase.

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There is no node  $j$  with  $e(j) \leq -4$ .

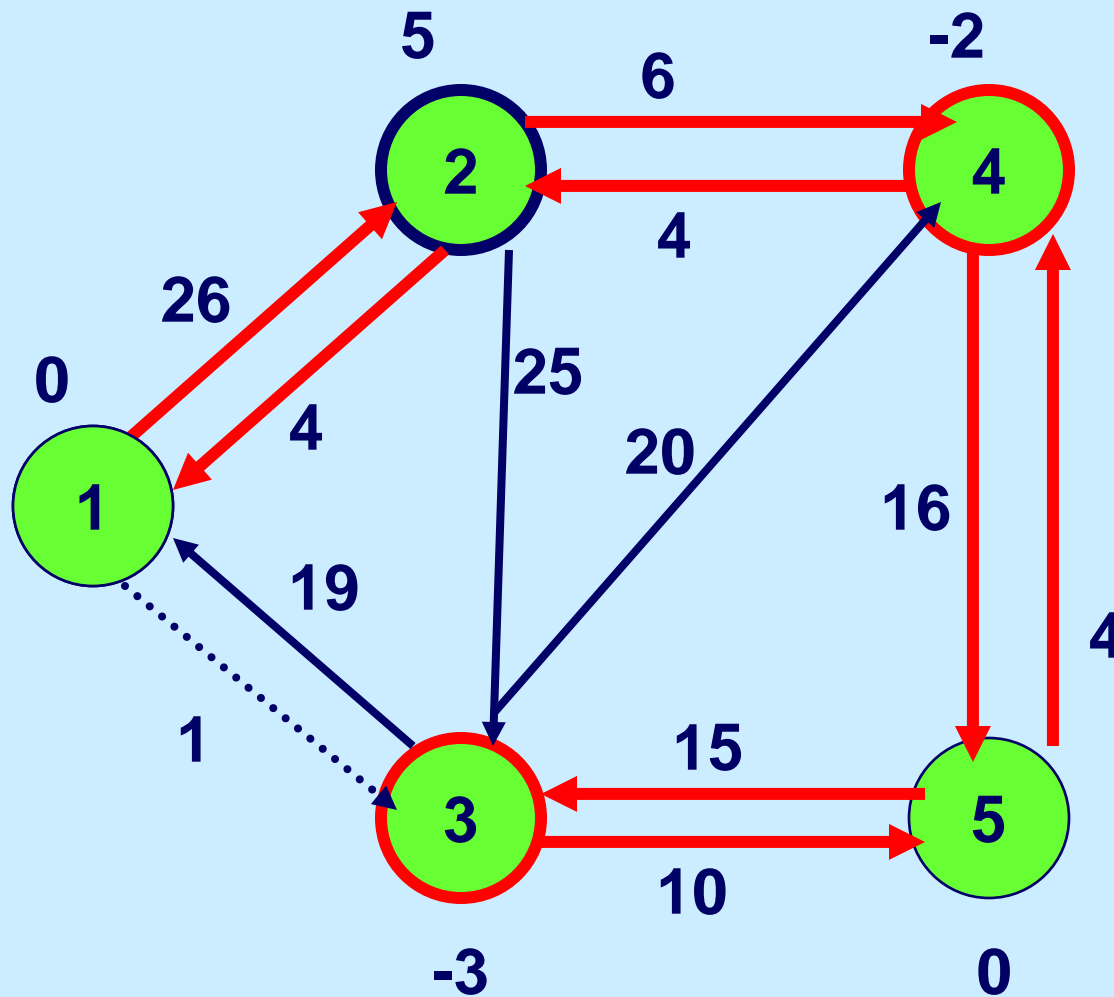
# Begin the 2-scaling phase



There is no node  $j$  with  $e(j) \leq -4$ .

What would we do if there were arcs with capacity at least 4 and negative reduced cost?

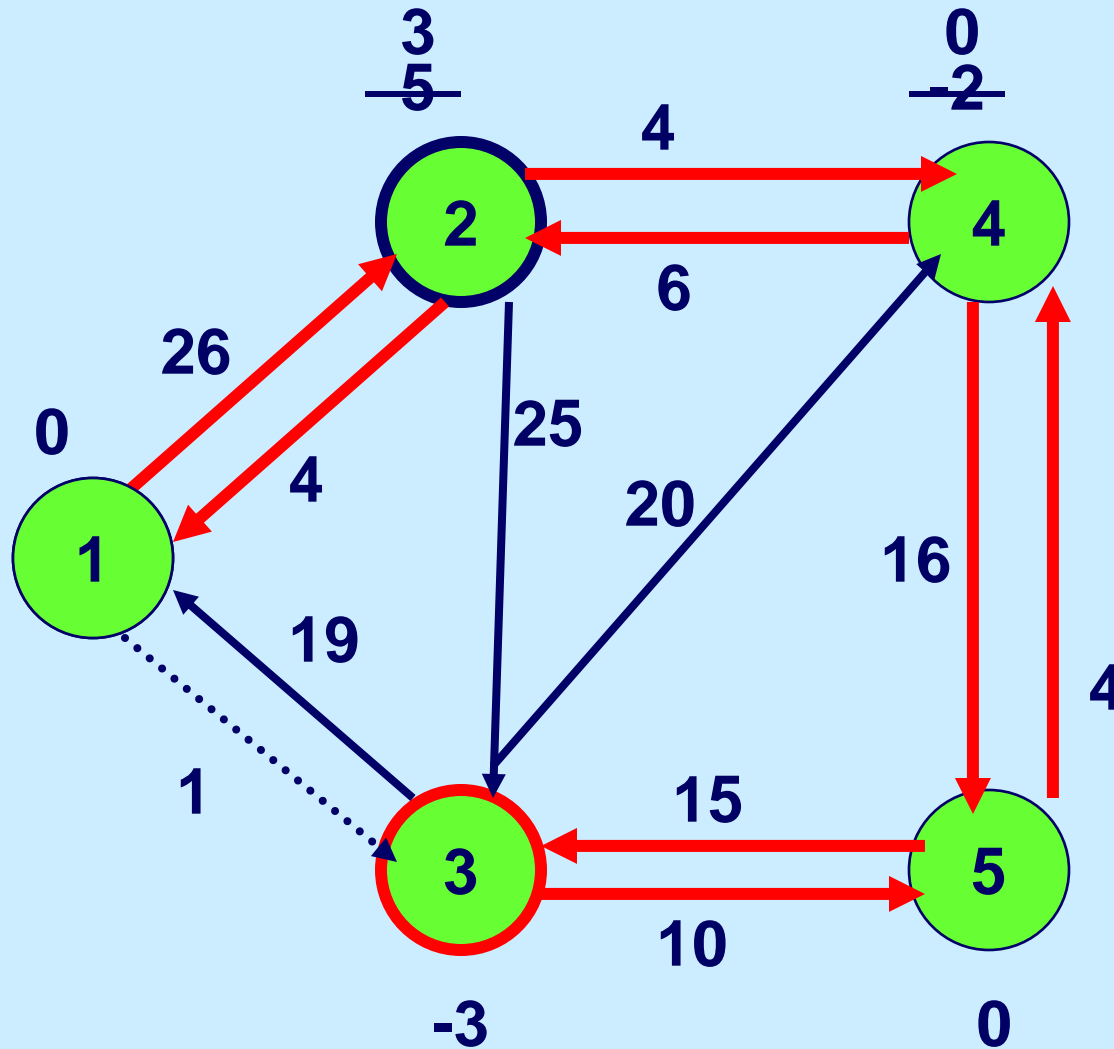
# Send flow along a shortest path



Send flow  
from node  
2 to node 4

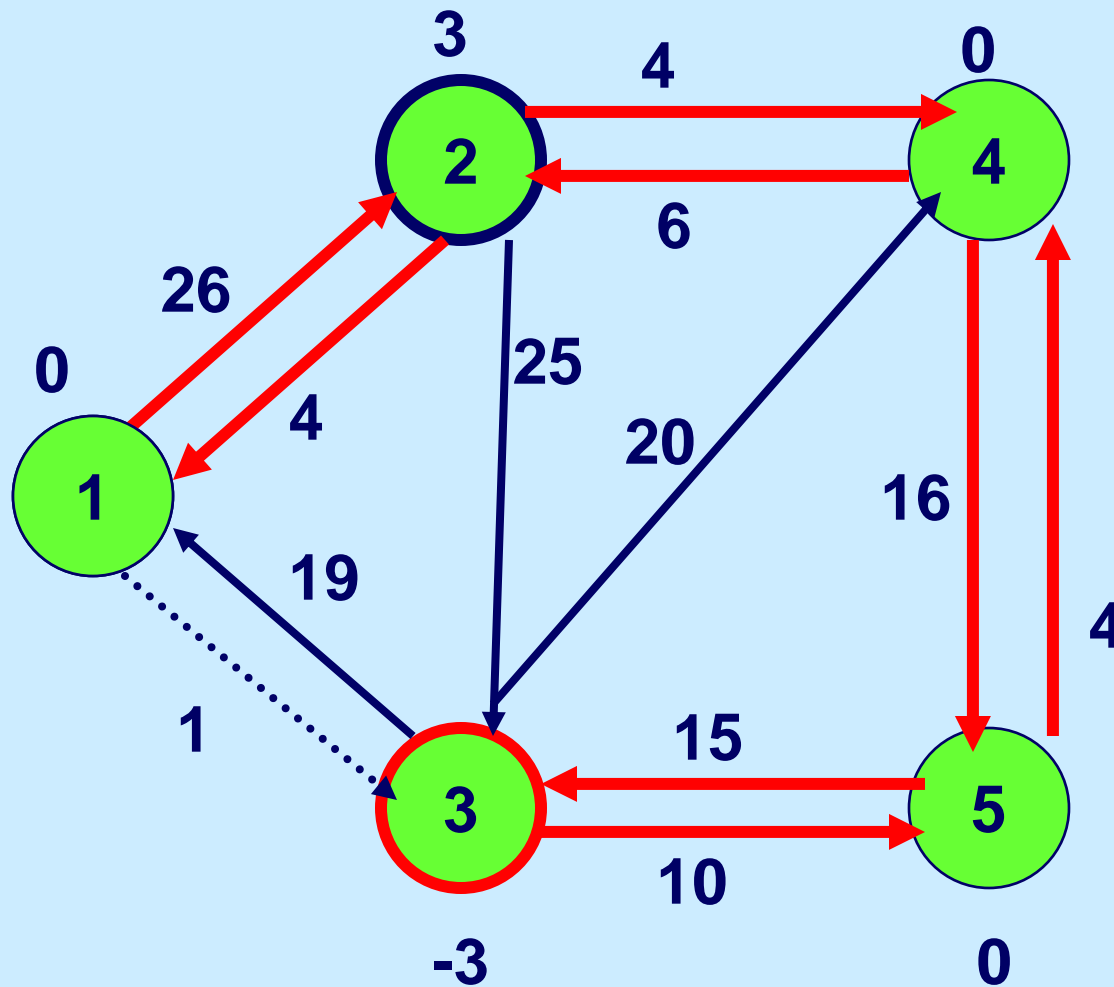
How much  
flow should  
be sent?

# Update the Residual Network



2 units of flow were sent from node 2 to node 4

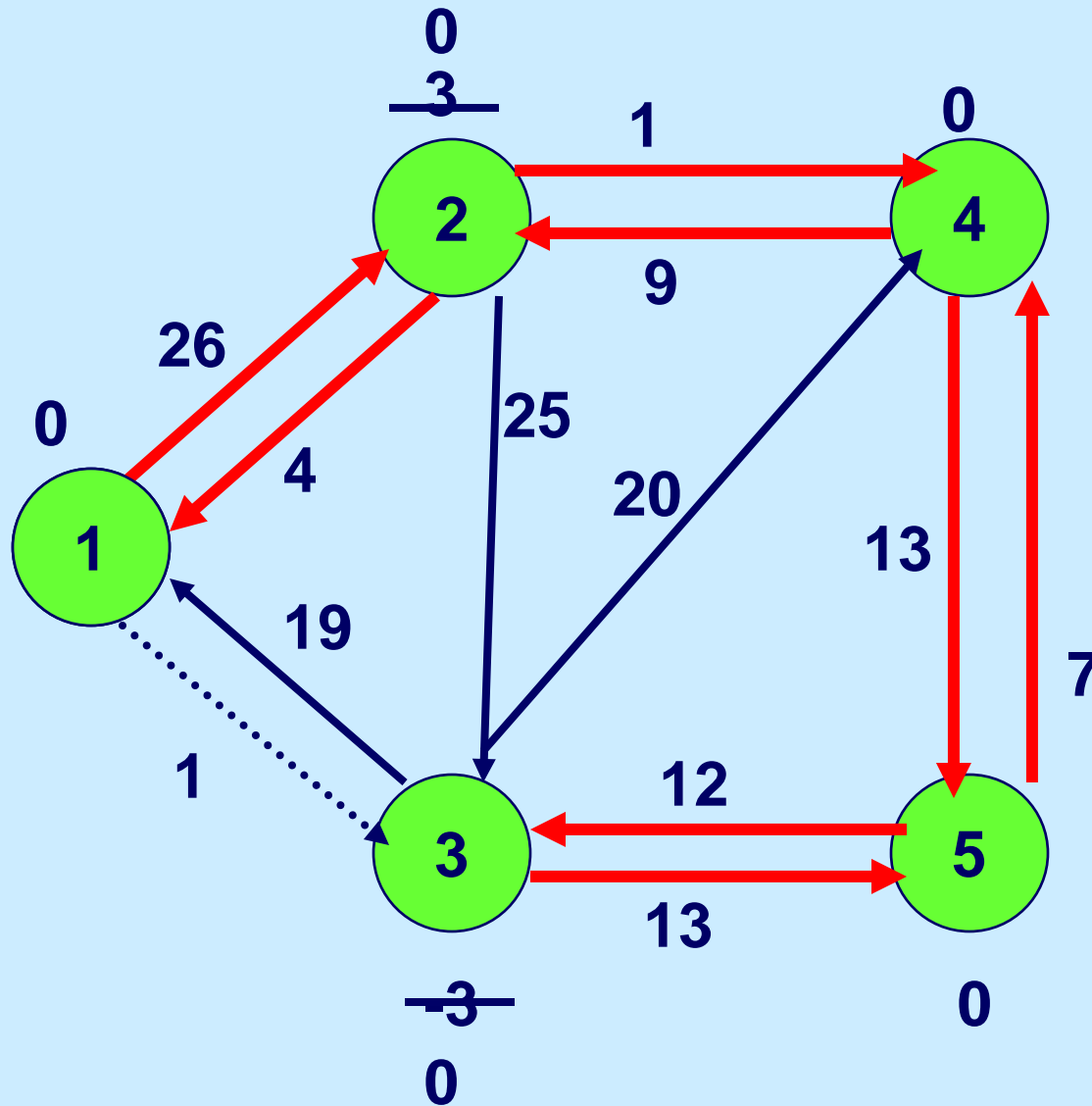
# Send Flow Along a Shortest Path



Send flow from node 2 to node 3

How much flow should be sent?

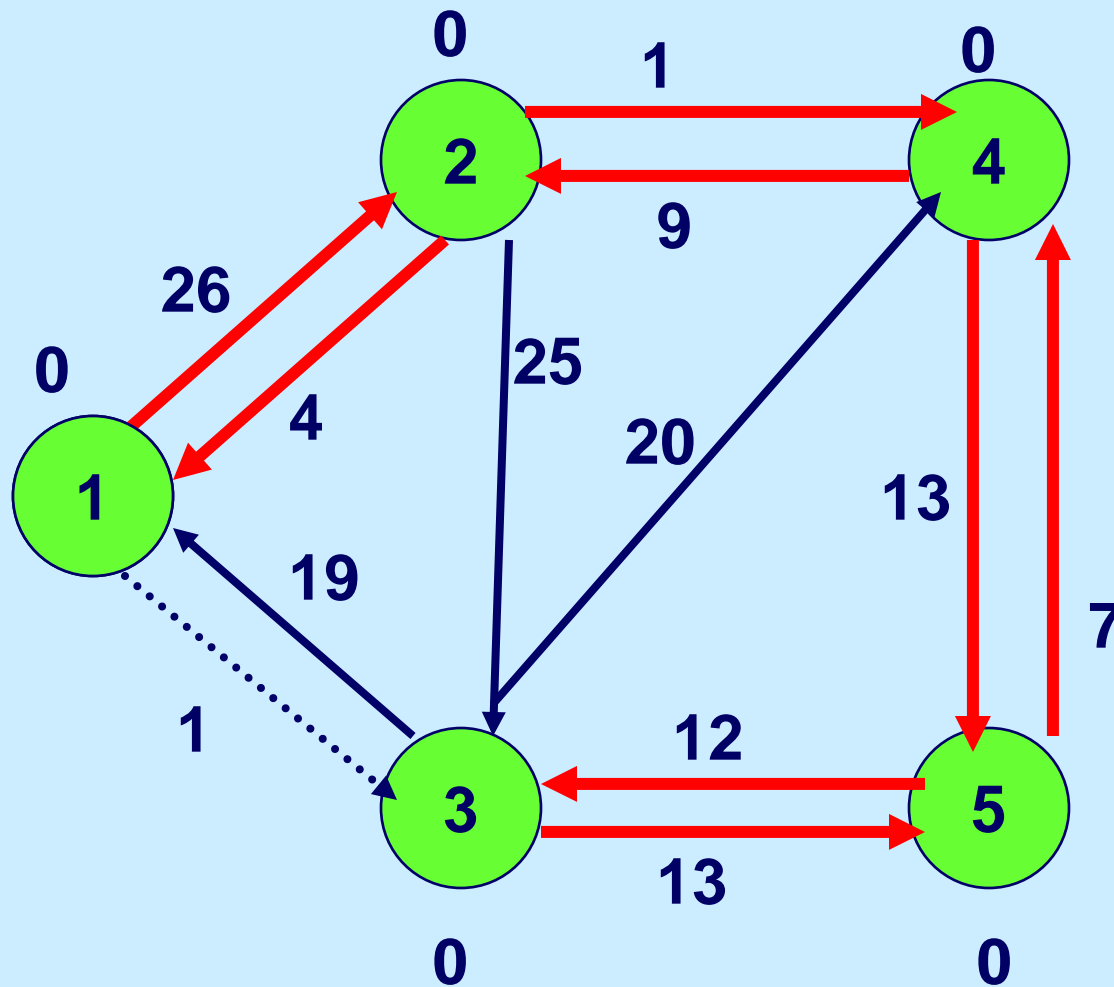
# Update the Residual Network



3 units of flow were sent from node 2 to node 3

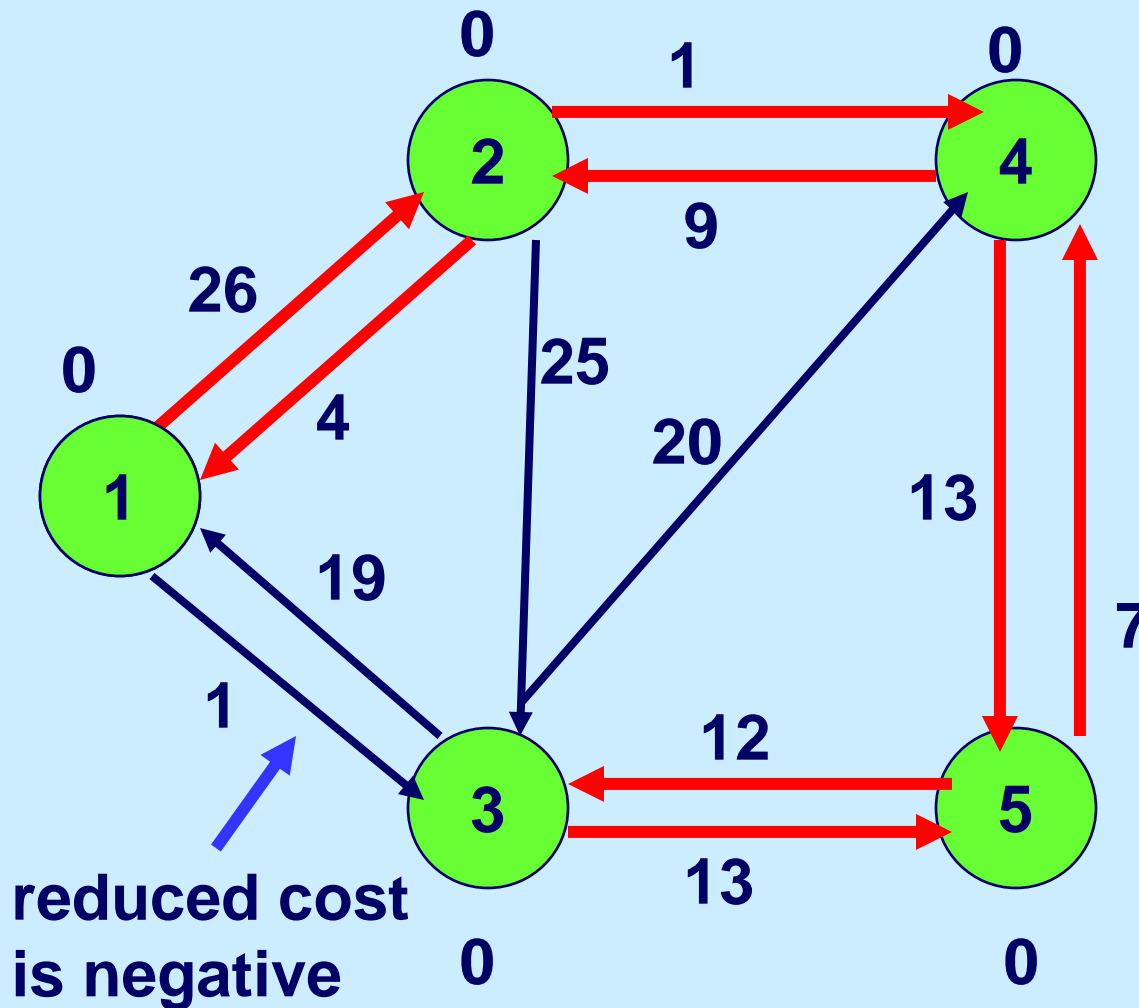
# This ends the 2-scaling phase.

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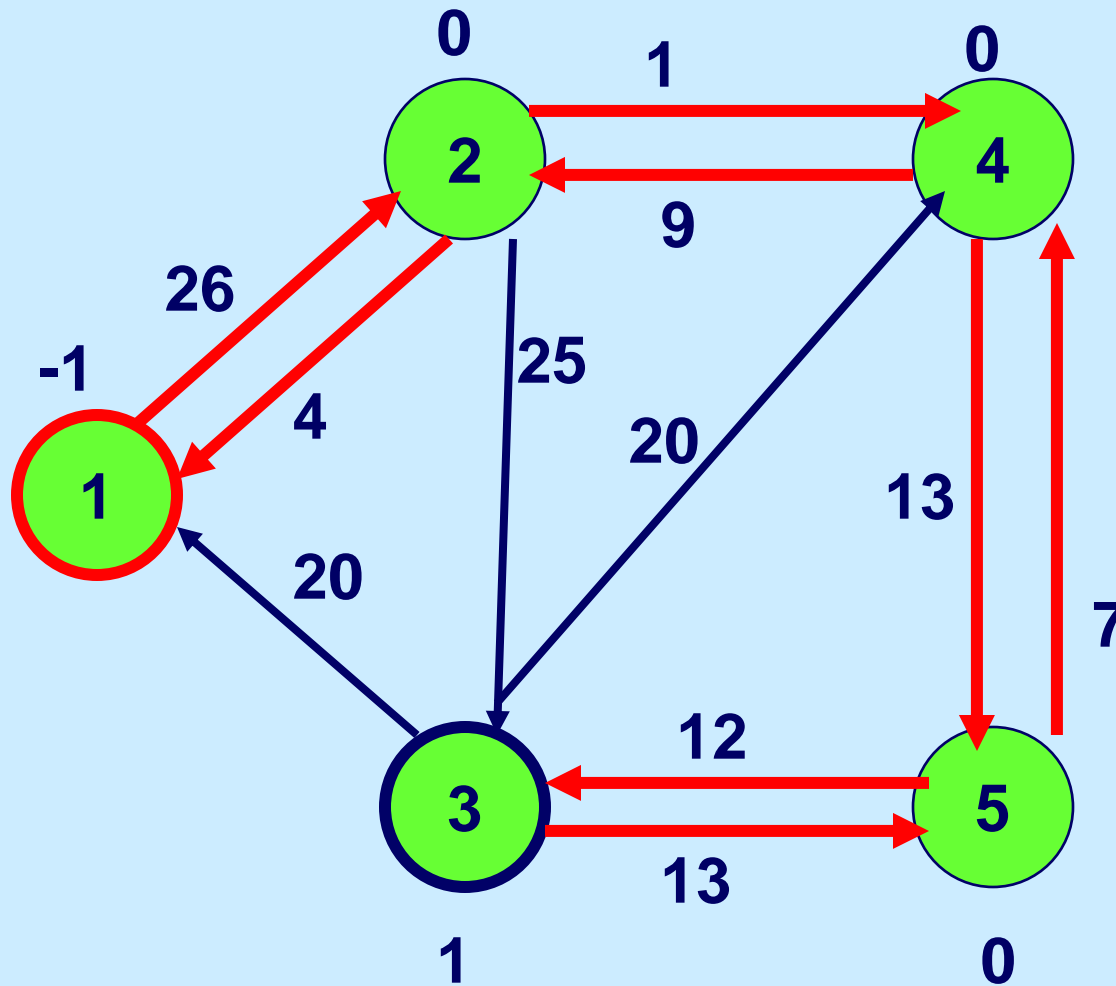
Are we optimal?

# Begin the 1-scaling phase.



Saturate any arc whose capacity is at least 1 and with negative reduced cost.

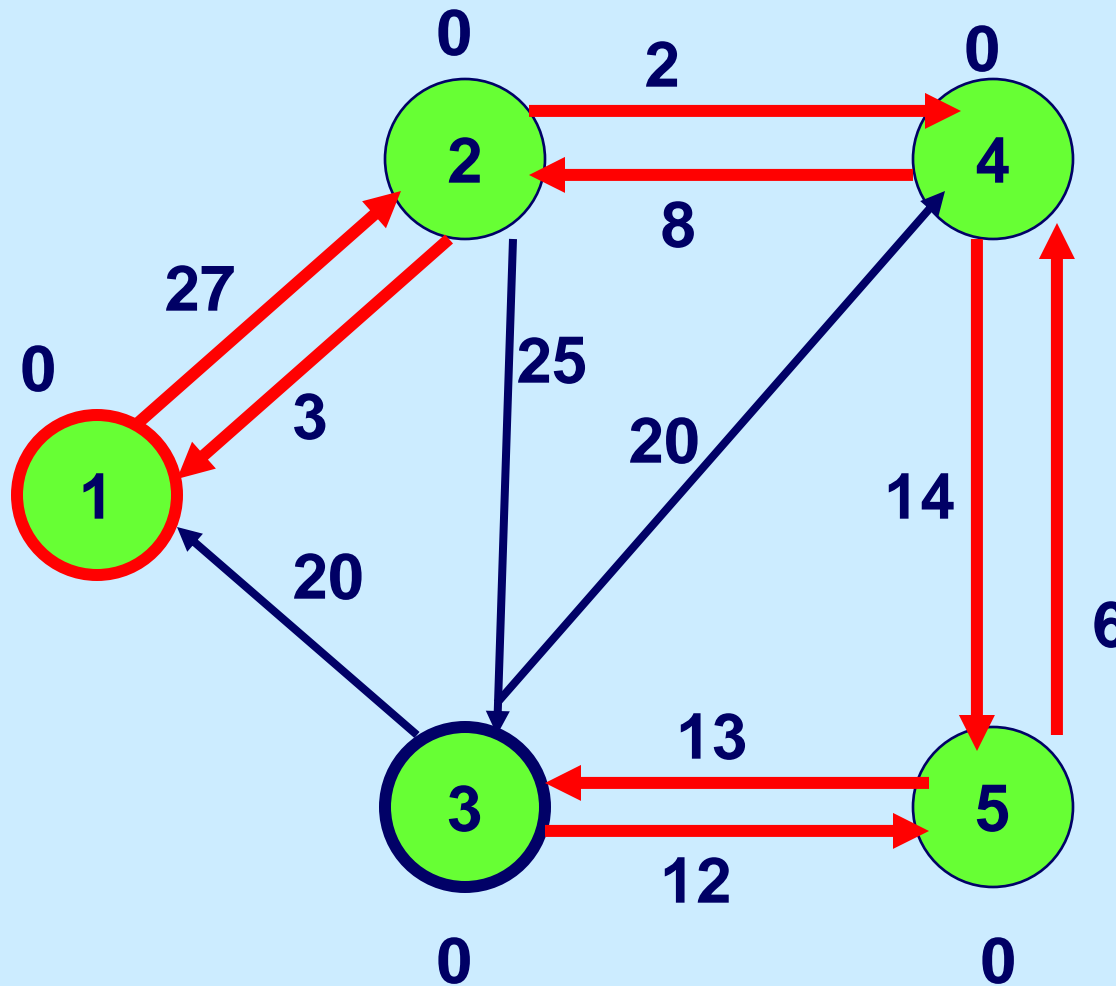
# Update the Residual Network



Send flow from node 3 to node 1.

Note: Node 1 is now a node with deficit

# Update the Residual Network

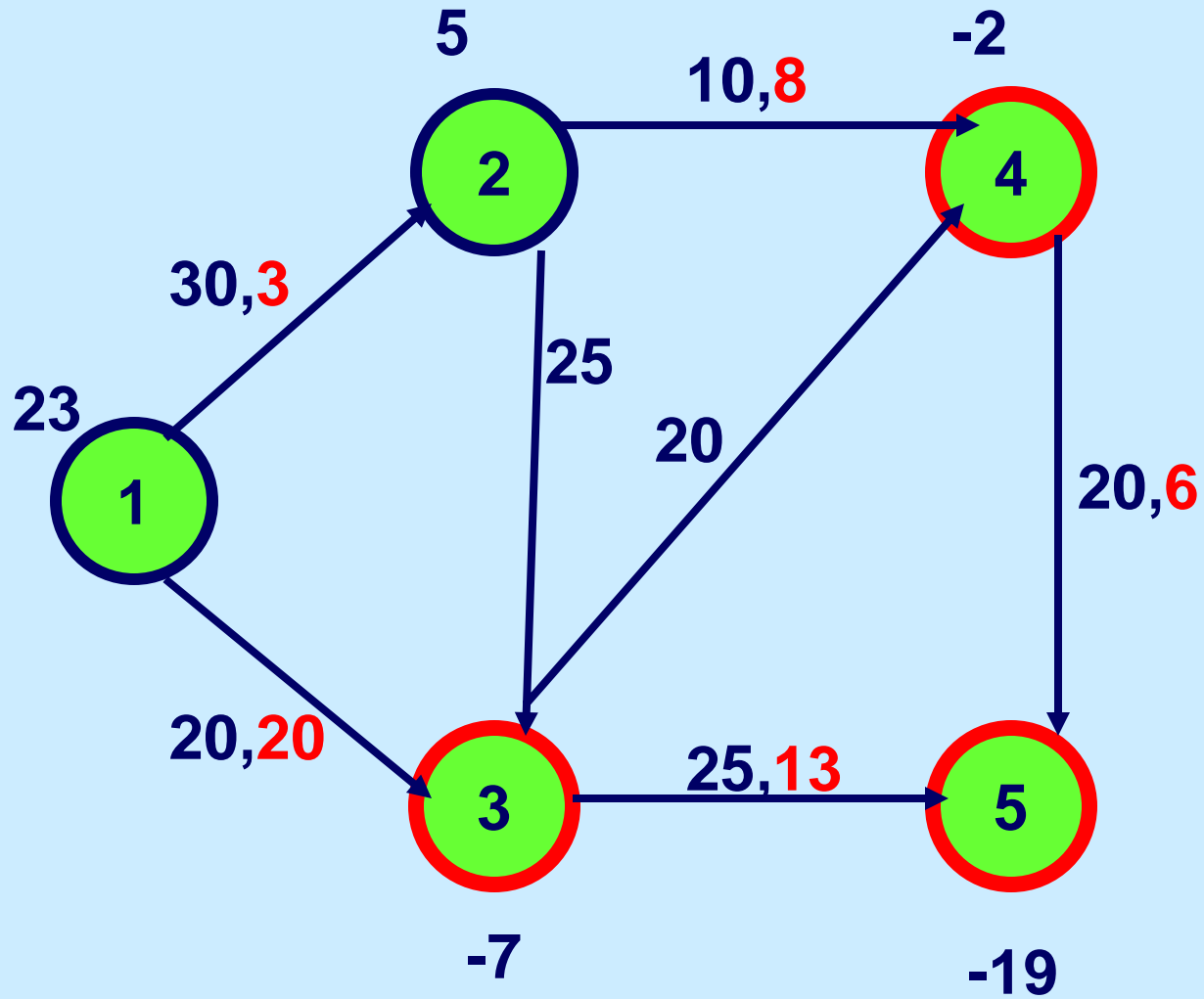


1 unit of flow was sent from node 3 to node 1.

Is this flow optimal?

# The Final Optimal Flow

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# The Final Optimal Node Potentials and the Reduced Costs

